REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

LIE SYMMETRY ANALYSIS OF THE COUPLED SYSTEM OF SINE-GORDON EQUATIONS

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A thesis submitted by Melis MENDİ in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE** is approved by the committee on 06.08.2021 in Department of Mathematics, Program of Mathematics.

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Signature

Dedicated to my dear mother

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LIST OF SYMBOLS

$\xi(x)$	Infinitesimal of a Lie group
$\eta(x)$	Infinitesimal of a Lie group
$\tau(x)$	Infinitesimal of a Lie group that is prolonged to the space with derivatives
X	Infinitesimal operator (generator) of a Lie group
prX	Infinitesimal operator of a Lie group in prolonged space
L_k	k-dimensional Lie algebra
$\mathbb{R}^n$	The Euclidean space
∂x	The set of all first order partial derivatives of the dependent variable x
$\mathbb{R}$	The set of real numbers
x^*	Transformed variable under the operation of a Lie group

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ABSTRACT

Lie Symmetry analysis of the coupled system of sine-Gordon equations

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Symmetry group analysis is one of the methods we can use when finding new solutions. In this thesis, we applied this method to the coupled system of sine-Gordon.This problem is important in DNA (deoxyribonucleic acid) dynamics. First, the fundamental properties of Lie transformation groups are explained and then examples of the equations that we can apply this method are given. We have considered that Lie symmetry method, which we can apply to all differential equations. Also we can extend it to differential equations systems. Symmetry generators and the corresponding infinitesmals of the problem are studied. We used the extension of the infinitesimal operator (generator) and the prolongation formulas. However, we used the generator coordinates to find the point transformations. The Lie algebra and the coordinates of the operators played a very important role in obtaining new solutions. Several symmetry transformation groups of the problem are obtained.The invariant criterions are presented.

Keywords: Lie symmetry analysis, System of PDEs , Group transformations,Non-linear equations

ÖZET

Sinüs-Gordon denklemlerinin birleştirilmiş sisteminin Lie Simetri analizi

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Simetri grup analizi, yeni çözümler bulurken kullanabileceğimiz yöntemlerden biridir. Bu tezde, bu yöntemi sinüs-Gordon'un birleşik sistemine uyguladık.Bu problem DNA (deoksiribonükleik asit) dinamiğinde önemlidir. Öncelikle Lie dönüşüm gruplarının temel özellikleri anlatıldıktan sonra bu yöntemi uygulayabileceğimiz denklemlere örnekler verilmiştir. Tüm diferansiyel denklemlere uygulayabileceğimiz Lie simetri yönteminin diferansiyel denklem sistemlerine kadar genişletilebileceğini gözlemledik. Simetri üreteçleri ve problemin karşılık gelen sonsuz küçükleri inceledik. Sonsuz küçük operatörün (üreteç) uzanımını ve uzanım formüllerini kullandık. Bununla birlikte, nokta dönüşümlerini bulmak için üretecin koordinatlarını kullandık. Lie cebiri ve operatörlerin koordinatları yeni çözümler elde etmede çok önemli bir rol oynadı. Problemin çeşitli simetri dönüşüm grupları elde edildi ve değişmezlik kriterleri sunuldu.

Anahtar Kelimeler: Lie simetri analizi, Kısmi Diferansiyel Denklem Sistemi, Lie nokta dönüşümleri, Lineer olmayan denklemler

YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

1.1 Literature Overview

For many years, people have observed the magnificent effects of symmetry in nature, art and design. When we think about natural phenomena, we see that symmetry supports us in understanding the laws of physics and formulating these rules. Most of the processes in our universe are reversible at the time, in other words, the laws of physics work the same back and forth on that event. It has been said that at the level of elementary particles, there is a physical process that appears differently when moving forward and backward.

Physicists have tested experiments. These experiments show that some particles directly break time symmetry. For instance, whenever a pair of quarks are brought together by strong forces. Occasionally, two different sequences take place and thanks to the weak force, they oscillate between these two sequences. But moving in a certain direction takes longer than going back. So if you could record this event, the forward stream would look different from the backward playback. This is exactly what breaking time symmetry means. That is, in certain situations elementary particles can show the distinction of moving forward or backward in time. Emmy Noether, a mathematician, prepared one of the most important works of the 20th century on symmetry. This work has been a very important guide in 20th and 21st century physics. Advances made in atomic and subatomic physics have shown that the microaleme is equipped with symmetry laws. By developing the Noether theory, she gave a big perspective to the new laws of physics. Researchers conducting research on particle physics continue to work from Noether's point of view.

An example of such an operation is "translation". It would be very helpful to give an example from Feynman's research. Let an experiment be conducted in a region of space. Then, let all the conditions that may affect this experiment and experiment be carried to another part of space and done again. If the results of both experiments are the same, this experiment is symmetrical with respect to the translation process

in space. If the same experiment is performed at another time without changing the location of the experiment in space and the same result is obtained, the experiment is symmetrical with respect to the translational process in time [1].

Symmetry is a concept in mathematics, especially geometry. For instance reflection symmetry, rotational symmetry, translational symmetry, helical symmetry are types of symmetry. Peter Hydon expresses symmetry of geometrical figure is a transformation. And its action leaves the object same in his book [2].

We can draw a line through a equilateral triangle to highlight its symmetry. If the shape is reflected through this line, then we obtain the same triangle, unmoved. If we apply a transformation of 120 degrees, we see that the appearance of the triangle does not change. This shows us that the transformation is symmetry. Also 240 and 360 degrees are symmetries for equilateral triangle. But we cannot observe anything for 360 degrees. We say for that trivial symmetry. One of the crucial and beautiful themes unifying many branches of modern mathematics.. It has been considered whether some shapes are more symmetrical. For example, a square is more symmetrical than a rectangle, and this tells us that a square is more symmetrical than any four-sided shape. Examples of symmetry for two-dimensional geometric shapes include mirror reflection. We can observe this example if we draw a line and project the shape without changing the view. Equilateral triangle has both symmetries (rotation and reflection) so symmetries are not special.

Symmetry exists not only in geometry but also in other fields of mathematics (which we will study on differential equations in this thesis). Group theory is the mathematical study of symmetry. It aims to find generalized rules. We will go into the details of symmetry in mathematics and express how studies have been done in applied mathematics.

Lie theory is also a branch of group theory. We can think of group theory as bringing together many of the topics we investigate like sets, cardinality, number theory, isomorphism, and modular arithmetic. It is almost impossible to find first order, homogeneous and linear equations in nature. If we could find it anyway, the concept of symmetry would not be needed for differential equations. Every problem would be very easy to solve. In order to find solutions to real-life problems, scientists have turned the problems into a mathematical model and searched for solutions. Learning and applying the method of symmetry means finding another equivalent solution for these challenging models. If we can achieve this under the invariance criteria, we can reach the solutions we are looking for. Invariants of a Lie symmetry group is very important. Lie symmetry group can be determined using the infinitesimal

operators of the Lie symmetry group, solving a semi-linear characteristic system of differential equations that are obtained from the operators [3]. We investigate how we can express the solution of a complex equation with another equation. Because we cannot reach numerical solutions of several inhomogeneous systems of equations. Keeping structure of the equation (under the conditions we will mention below) we obtain transformations between another equation. The structure is slightly different when making these changes. Different objects represent different equations. We will elaborate on how the dependent and independent variables behave within these equations.

Symmetry has a important role in differential geometry, applied mathematics, numerical analysis, abstract algebra and analysis. Particularly, in the study of symmetries in mathematics, our objects are differential equations and our transformations are point symmetries, contact symmetries, generalized symmetries, etc. It is referred to as invariance and means that a mathematical object remains the same under transformation, mapping of any structure onto itself. There is a possibility of realization in many ways [4, 5]. Lie's research is so extensive that it has aspects that touch geometry, algebra and applied mathematics. When symmetry analysis is applied to mathematics, we obtain very useful information about equations. Such as being able to reduction, integrate, linearize, obtain generalized rules. If the vector space formed by the symmetries admited by the diferental equaions forms the Lie algebra. We reduce the order of the differential equation to the order of Lie algebra. Symmetry is a functional tool for solving differential equations. We could say that it is the most general method to reach exact solutions.

In this thesis we will study Lie symmetry groups of the ordinary (ODE) and the partial (PDE) differential equations. We will investigate symmetry transformations and obtain invariance criterions for the corresponding Lie symmetry transformations. We investigate symmetry transformations of the system of sine-Gordon equations. It is related with the methaematical modelling of DNA (deoxyribonucleic acid) dynamics.

1.2 Objective of the Thesis

There are many ways to solve differential equations. Studying solution methods for differential equations is very important. We often see the differential equations in many fields of science like molecular biology and genetics, physics, mechanics, engineering economics, e.t.c. One of these methods is Lie symmetry analysis.

For years, scientists have tried to comprehend complex natural phenomena and try to turn these into mathematical modeling. Lie symmetry analysis is also one

of the significant method and used in the solution of these models. Complex natural phenomena are transformed into differential equations. Many of these are nonlinear and inhomogeneous differential equations. Thanks to symmetry analysis in mathematics, we can get some information about equations like integrability, linearization, classification. Symmetry analysis also provides a wide range of possibilities for linear partial differential equations, but systems of linear equations lead us to conclusions for the approach in the first step of the problems in physics and mechanics.

The methods by which we solve linear equation systems, like Fourier method and Laplace transform method cannot be applied to non-linear equations. It has not been possible to build just one method for all differential equations. The strongest aspect of Lie symmetry analysis is that we can apply it to almost any differential equation e.g ordinary or partial, linear or nonlinear, constant or variable coefficient. When we think of methods other than Lie symmetry analysis, most of them have the condition of being integrable or other conditions. The symmetry method is like a generalized method that includes many methods. For example separation of variables method and the method of integrating factor, are a case of this method. We consider symmetry as a transformation. This transformation transforms any solution to equivalent solution.

In Lie's studies transformations are like groups that depend on continuous parameters and consist of either point transformations. These transformations act on the system's space of independent and dependent variables. We can say that contact transformations acting on the space of independent and dependent variables as well as on all first derivatives of the dependent variables. Translations, rotations, and scalings are basic examples for Lie groups.

The symmetry analysis of the differential equation, which was discovered by the Norwegian mathematician M. Sophus Lie, is a theory that enables new solutions of differential equations. Symmetry analysis of the differential equation is a theory that systematically created by finding local transformation groups. They leave differential equations invariant.

We briefly give some informations about Sophus Lie and process of Lie group analysis to understand Lie's studies. In the early 1800's young mathematicians Évariste Galois and Niels Henrik Abel wanted to obtain equations with solutions corresponding to higher order equations. An algebraic equation with reduced order having a solution set of functions of the corresponding higher order equation was wanted to be found. Galois discovered a method to solve algebraic equations.(which is called Galois theory now.) This method of Galois shed light on the beginning of Sophus Lie's work, he

took lectures about Galois' studies. Then he studied a forceful method for solving differential equations like algebraic equations with Felix Klein. Klein has an important place in Lie's life and academic career. Before turning to different fields, they worked on geometry, symmetry, and group theory together for a while.

If we need to establish a similarity between Galois and Lie, we can say that just as there are polynomials whose roots are investigated in Galois theory, there are higher order differential equations whose solutions are investigated in Lie's studies. We can see Lie's results as a far reaching generalization of the Galois's and Abel's theory of solubility of algebraic equations by radicals [6].

Lie actualized his study about groups of transformations when he was professor in University of Christiania and he determined the idea of Lie group analysis.

He was actually continuing his studies in the field of geometry before. He dealt with line geometry, to explain a little more, with problems investigating the motion of any transformation in the set of transformations one surface into another. French mathematician Jean Gaston Darboux (1842-1917) influenced him with his studies which associate the subjects differential geometry and differential equations. Lie studied on homogeneous curves invariant under a group and he named it W-Curves with Klein. It is curve that is invariant under group of projective transformations. Projective transformations are provided him to discover contact (tangent) transformations, that map surfaces in a space including points and their tangents. These studies led to the formation of Lie groups. Then he studied contact transformations to group transformations and stability groups of differential equations for integrating the equations. In Lie symmetry analysis, transformations of a Lie group move the solutions of the corresponding differential equation to the solutions we are looking for. And at the same time, as in Galois' theory, there is a requirement for solvability for differential equations.

We usually apply the symmetry method to get exact solutions of differential equations. We use it for ODEs to decrease the order of the higher order differential equations. If we think for partial differential equations, we can reduce the number of independent variables and this can make the equation become ordinary differential equation. Symmetry groups leading to invariable solutions of boundary value problems. Systematic procedure can reach new solutions, as well as ordinary differential order of the equation, partial differential reducing the number of variables of the equations and converting to an ODE. For instance, n be the number of independent variables of a PDE if the equation admits (r is less than or equal to n) r-parameter symmetry group we can decrease these variables to r. From the solution of the differential equation

we have reduced. We obtain the solution of our main equation. We will give some examples next chapters like translations, rotations, scalings.

Lie struck on that the notion of infinitesimal operator, in other word generator include all existing integration theories. But Lie's studies was not used on the differential equations for a while.

In fact, it took some time to realize the wideness of the Lie symmetry method. Russian mathematician Ovsiannikov has worked hard on the implementation of the symmetry method to differential equations. Systematically applied to complex nonlinear equations. Nail H. Ibragimov(1939-2018), who has written books in both mathematics and mathematical physics, has also worked in this area [3, 7]. Over time, Lie symmetry method studies have been carried out by many scientists. Since 1989, many studies have been examined and articles, thesis and books have been written. The method supplies a wide approach to solve differential equations converting complicated non-linear conditions to simple linear ones. Thus, symmetry method become one of the most common tools.

While trying to reach the solution of the differential equation that admits the Lie group, we are actually applying an algorithm. We can specify the steps like this; writing determining equations, separating these equations with respect to arbitrary elements and solving these equations [3, 7]. By applying this algorithm, we map the solution of one equation to the solution of another equation and obtain new exact solutions. First, we explained the solution method of symmetry for ODEs. Sometimes, thanks to this method, we were able to get integrals comfortably and sometimes we were able to reduce orders. For PDEs, the situation is similar. Sometimes a PDE can transform into an ODE, sometimes it cannot.

1.3 Hypothesis

In order to practice Lie symmetry method on system of second-order, non-linear partial differential equations in the multidimensional case we focus on coupled system of the sine-Gordon equations. Edmond Bour (1832-1866) discovered sine-Gordon equations in the course of study. Scientist interested with the equation in the 1970s because of its soliton and multisoliton solutions.

This system consists non-linear hyperbolic PDEs with the sine of the unknown functions and the problem is important in DNA (deoxyribonucleic acid) dynamics. We seek another equivalent solution for a difficult to solve inhomogeneous system of equations.

$$\begin{cases} u_{tt} - u_{xx} = \sin(u - v), \\ v_{tt} - v_{xx} = \sin(u - v). \end{cases}$$

The reason we want to work with such an equation system is because it contains some kind of wave equation and there are similar examples in many branches of science in nature, for example physics, mechanics, general relativity. We have seen application of the method to physical and engineering PDE systems in books published by scientists such as G. Bluman and S. Anco.

Obtaining transformation groups which leave the solutions of differential equations invariant is one of the most effective systematic tools in the analysis of non-linear equations. Analytical solution methods are quite limited. By using these transformation groups and associated symmetry algebras, it is possible to reduce the number of variables of a PDE and find complete solutions by reducing it to an ODE if there is a symmetry algebra. Classification of differential equations by taking symmetry groups into account is an interesting problem in terms of mathematics, as it provides the opportunity to distinguish the equation families with certain symmetry algebras belonging to a general class of equations by determining them. The obtained algebras also provide guidance for group-invariant solutions of the representative equations of these families.

Applied mathematicians argue that the symmetry method is an significant method [2]. We can solve all ordinary or nonlinear ordinary differential equations by symmetry transformations.

In the first part of this thesis, a result is obtained by applying symmetry transformation to a nonlinear system of ordinary differential equation whose general solution is given. We follow the path below for obtaining Lie point symmetries acting on the system of partial differential equations. We tried to present fundamental theory of Lie groups and explain how we use Lie symmetry method in differential equations. Symmetry analysis is a very broad concept. In this thesis, we searched for equivalent differential equations. Equivalent equations are equations that have been converted to each other with point transformations. We illustrate symmetries as transformations on the set of solutions of a differential equation and we can think integration constants as parameter of the admitted symmetry group. The reduction of the nonlinear sine-Gordon equation system by the symmetry method is made. While doing this reduction, one-dimensional subalgebras of Lie algebra belonging to Lie group which leave the equation symmetric are used. New reductions can be made using other subalgebras.

First, the definitions of Lie group and Lie group transformations and their infinitesimal operator are explained. The First Fundamental Theorem of Lie showed us that the Lie group of one-parameter transformations correspond to an initial value problem with a first order ODE. [8]. We clarified the extension of infinitesimal operators of the point transformations. Infinitesimal operators with coordinates of dependent and independent variables are used to determine the point symmetries. We presented the prolongation formulas that we used. We applied the prolongation formula to the equations in the system. This process leads us to find infinitesimals. We have obtained the auxiliary function by expanding the source functions to power series and we have obtained the bases of the Lie algebra space. We obtained bases, generators and Lie formulas.

We expressed that we can extend the prolongation formulas to the partial differential equations and their systems. We demonstrate how to obtain the determining equation by using prolongation formulas. Also the invariance criterions are presented. We classified the equations of the Lie symmetries, to understand what it means to apply Lie symmetry method to equations under invariance criteria. We show how we find the point symmetries and related invariant solutions. Also we can find group invariant solutions and these invariant solutions satisfy us to characterize the differential equations by linearity, integrability or solvability.

Lie symmetries helped us to examine these equations in terms of linearization and integration, as well as classification by having the same properties. The set of point symmetries of the differential equation are detected. It is possible to demonstrate the infinitesimal operator of the group admitted by a given partial differential equation. This generator of a base of Lie algebra. This corresponds to the Lie symmetry group of the equation is a linear combination of this base with any element. Invariant solutions can be obtained. Any element of this base since the algebra is a one-dimensional subalgebra, this invariant solution is of all invariant solutions family created with subalgebras. It can correspond with any transformation to another invariant solution in the set.

Our main objectives are fundamental theories of Lie groups. Afterwards its application to the coupled system of sine-Gordon equation are considered. It generalizes the viewpoint to the system of PDEs.

Our thesis consists of 4 chapters and prepared as follows: The second chapter is about fundamental theory of Lie groups and explains how Lie symmetry method is used in differential equations. The third chapter investigates our main results which consist of applications of Lie symmetry method to the coupled system of the sine-Gordon

equations. It generalizes some approaches and definitions to the case of PDE systems. The fourth (last) chapter consists of results and discussion of the study.

2
THE SYMMETRY METHOD OF DIFFERENTIAL EQUATIONS

We can apply the symmetry method to all differential equations. We will apply the symmetry method for both ODEs and PDEs. In other words, we will work on mathematical models that we can express with differential equations. Lie group and group transformations help us to understand this method, which we can apply to differential equations. Lie group theory is a branch of group theory discovered by Galois for algebraic equations. We can use the symmetry method for integration, linearization or general solutions of differential equations. It is a forceful method.

Throughout this study, we will be interested in solving differential equations. When real life problems are considered, higher order derivatives will be seen. Dimensional analysis provides us when analyzing the differences between physical quantities. Dimensional analysis supports us in order reduction operations. For example, it allows us to perform a reduction operation in a PDE system expressed as a BVP [9].

The transformations form the Lie group. These transformations are solutions that lead one solution to another. In other words, it is these transformations that provide other solutions that satisfy the equation. Variables and space of variables are considered when classifying groups. The variables of the corresponding differential equation (dependent or independent) are affected by a Lie group of point transformations. In other words all variables and first derivatives of the dependent variables are affected a Lie group.

The key point in symmetry analysis studies is infinitesimals and infinitesimal generators. An infinitesimal generator is required to apply it to a differential equation. We will explain about how to obtain and use this infinitesimal generator with Lie's algorithm. In order to act on the derivative terms, we demonstrate the extension of this infinitesimal generator. Our main equation may not be linear or homogeneous. The determining equations are linear and homogeneous, and we will solve these

equations. Considering the multi-parameter Lie groups, a Lie algebra is formed for the infinitesimal operators corresponding to each parameter.

2.1 Lie Group Theory

2.1.1 Symmetry Transformations

We mentioned in the introduction that symmetry is actually a transformation and that we can call this transformation symmetry under some conditions. These conditions are; being diffeomorphism, preserving the structure and mapping the of the element to itself. As an example of a geometric shape, we have an equilateral triangle at the previous chapter. As Peter Hydon explains in his book these examples can be multiplied. For example, if we examine the symmetries of the unit circle:[2]

These 4 transformations that we have given as examples satisfy the conditions of symmetry. So the new x and y can be differentiable at any point, furthermore the inverse of φ_1,φ_2,φ_3,φ_4 also are the symmetry transformations. So it is clear that the transformations are diffeomorphisms and at the same time they maintain the form.

$$\varphi_1 : (x, y) \to (-x, y)$$

$$\varphi_2 : (x, y) \to (x, -y)$$

$$\varphi_3 : (x, y) \to (-x, -y)$$

$$\varphi_4 : (x, y) \to (x \cos a - y \sin a, y \sin a + x \cos a)$$

for each a $\in (-\pi, \pi]$. Here a will be the continuous parameter, which we will define later. We will study with the Lie group created by the transformations. When a equals to 0, φ_4 will be a trivial symmetry of the unit circle. Since trivial symmetries will not help to find another solution, we will not deal with them in our research. The transformations $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are symmetry transformations for the unit circle.

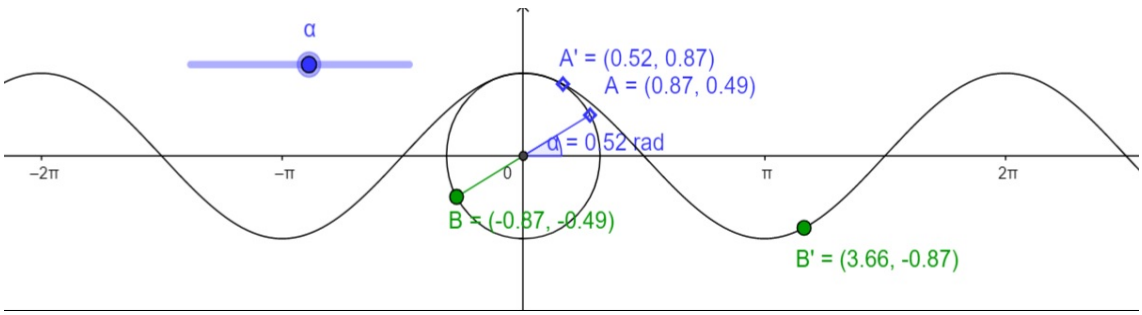


Figure 2.1 The unit circle

Examples of symmetry transformations in geometry can be multiplied by considering polar coordinates. We will focus more on the studies of symmetry in the field of applied mathematics.

2.1.2 Lie Group of Transformations

We mentioned that Lie group analysis is a branch of group theory. In order to understand Lie groups, we will first express the group definition. A group is the set in which an operation is defined that performs the following four axioms between its elements.

Definition 2.1 (Group)**.** If the set G defined by a φ operation (group operation) satisfies the following axioms. we can say G is a *group*. Here φ is a binary operation.

1. (*The Closure Property*) The result of any two elements under the operation φ, must be in the set. That is, for each $a, b \in G$:

$$\varphi(a, b) \in G.$$

2. (*Identity element*) A unique element that does not change anything when combine with any element. There is a identity element $e \in G$ such that

$$\varphi(a, e) = \varphi(e, a) = a.$$

3. (*Inverse element*) For any element of G the set involves a unique element a^{-1} such that

$$\varphi(a, a^{-1}) = \varphi(a^{-1}, a) = e.$$

4. (*The Associative Property*) The following equality is satisfied for any three elements of the set. For any $a, b, c \in G$:

$$\varphi(a, \varphi(b, c)) = \varphi(\varphi(a, b), c).$$

In addition, if a subset of G that defines a group (with the same binary operation φ) we call it is a *subgroup* of G.

Definition 2.2 (A Local Lie Transformation Group)**.** Let $x = (x_1, x_2, \cdots, x_n) \in D \subset 8\mathbb{R}^n$ and

$$x^* = X(x; a) \tag{2.1}$$

be the set of transformations for each $x \in D$ and parameter a $\in E \subset \mathbb{R}$, with $\varphi(a, b)$ defining a composition law of parameters $a, b \in E$.

1. (*Closure property*) For each $a \in E$, the transformations are one-to-one and onto.
2. (*Inverse element*) The set E with operation φ constitutes a group structure.
3. (*Identity element*) There is an identity element a_0 such that

$$X(x, a_0) = x \tag{2.2}$$

   for every element in D. That is, $x^* = x$ when $a = a_0$ is the identity element.
4. (*Associative property*) If $x^* = X(x; a)$ and $x^{**} = X(x^*; b)$, then

$$(x^*)^* = X(x; \varphi(a, b).$$

The above four conditions were sufficient for the group to become a one-parameter transformations group.

When the following conditions are satisfied, a one-parameter transformations group describes a local one-parameter continuous transformation group (a local Lie transformation group).

5. a is a continuous parameter. (E is an interval in $\mathbb{R}$.) when a=0 it forms the identity element.
6. X is infinitely differentiable with respect to $x \in D$ and it is an analytic function of $a \in E$.
7. $\varphi(a, b)$ is an analytic function of a and b, $a, b \in E$.

The analyticity of the group operation φ is a very important condition. Because Lie transform groups are parameterized by analytical operations. While doing our study on real numbers, this operation turns into ordinary sum.

Let's consider a few examples

- The translations group in the plane, $a \in \mathbb{R}$

$$x^* = x + a$$

$$y^* = y$$

- The group of scalings in the plane, $-1 < a < \infty$

$$x^* = (1+a)x$$

$$y^* = (1+a)^2 y$$

Here we observe that the identity element is $a = 0$.

- In the case of rotations:

(For example, we demonstrated the unit circle before.)

$$x^* = x\cos a - y\sin a$$

$$y^* = y\sin a + x\cos a$$

- The dilations on the real space, $-\infty < a < \infty$

$$x^* = e^a x$$

$$y^* = e^{2a} y$$

2.1.3 Lie Equations and Infinitesimal Operator

Consider the following definitions for the identity element $a = 0$ and one parameter Lie group transformations

$$x^* = X(x;a) \tag{2.3}$$

under the operation φ:

If we expand (2.3) to Taylor series around a=0, we get the following transformation. We call to (2.4) infinitesimal transformation.

$$x^* = x + a\xi(x) + O(a^2) \tag{2.4}$$

where

$$\xi(x) = \frac{\partial X(x;a)}{\partial a}|_{a=0}. \tag{2.5}$$

We name this equality *Lie equations,* it is a Initial Value Problem with initial condition $a = 0$. The components of $\xi(x)$ as *infinitesimals* of (2.3). These components are also referred to as *coordinates* in some resources.

We call *infinitesimal operator* of the Lie group to X.

$$X = X(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} \tag{2.6}$$

If we want to show with the two-components:

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \tag{2.7}$$

We find infinitesimals by solving *Lie equations*, which are ordinary differential equations in the first order by integrating. And so we will find the infinitesimal operator.

$$\frac{dx^*}{da} = \xi(x^*), \; x^*|_{a=0} = x. \tag{2.8}$$

In addition, we can also use a theorem to express Lie group transformations instead of using Taylor series expansion.

Theorem 2.1. *A one parameter Lie transformations group of form (2.3) with infinitesimal generator (2.1.3) is expressed in terms of a power series*

$$x^* = e^{aX} x = x + aXx + \frac{1}{2}(a^2) X^2 x + \cdots = \sum_{p=0}^{\infty} \frac{a^p}{p!} X^p x, \tag{2.9}$$

where the operator $X^p = X^p(x)$ is defined by $X^p = XX^{p-1}$, $p = 1, 2, \ldots$.

2.1.4 Invariant Functions

Definition 2.3. Let $f(x)$ be a locally analytic function and $f(x) \neq 0$. If the following equality is satisfied for each transformation of the group, f is called *group invariant*.

$$f(x^*) = f(x). \tag{2.10}$$

With the following theorem, we will see how to use the definition of invariant with the infinitesimal operator. Since this invariant theorem is a necessary and sufficient condition, it is a method that we can use as a definition. It is the most important rule we will use when applying the Lie algorithm to the differential equations.

Theorem 2.2. *With X, the infinitesimal operator of the Lie group of transformations*

(2.3), for a function $f(x)$ to be group invariant necessary and sufficient condition is

$$Xf(x) = 0. \tag{2.11}$$

This equality (2.11) is called invariant condition of $f(x)$ under a Lie group of transformations. (2.3).

We will give examples of invariants of Lie group transformations. We will focus on point transformations that can be used interchangeably under the condition of invariance.

Example. The Lorentz transformations in two dimensional space is given by

$$x^* = xcosha + ysinhay,\ y^* = xsinha + ycosha.$$

With this example, we show how to find coordinates (infinitesmals) using Lie equations and how to obtain the infinitesimal operator X.

$$\xi(x,y) = \frac{\partial(xcosha + ysinhay)}{\partial a}|_{a=0} = y \tag{2.12}$$

$$\eta(x,y) = \frac{\partial(xsinha + ycosha)}{\partial a}|_{a=0} = x \tag{2.13}$$

Therefore, the infinitesimal operator X takes the form:

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

2.1.5 Canonical Coordinates

Canonical variables act as a kind of variable replacement and move an element of the group, allowing one to write the given Lie group transformations and the infinitesimal generator in a simpler way.

Theorem 2.3. *Every one-parameter Lie transformations group (2.3) with the infinitesimal operator X can be changed with the canonical variables*

$$y_i{}^* = y_i,\ i = 1,2,\ldots,n-1, \tag{2.14}$$

$$y_n{}^* = y_n + a, \tag{2.15}$$

with the new infinitesimal operator (generator)

$$Y = \frac{\partial}{\partial y_n},$$

and the new coordinates $y = (y_1, y_2, \ldots, y_n)$ *called canonical coordinates and satisfied the following first-order linear differential equations:*

$$Xy_i(x) = 0,\ i = 1, 2, \ldots, n-1, \tag{2.16}$$

$$Xy_n(x) = 1. \tag{2.17}$$

Example. [8] Review the group of rotations

$$x^* = x\cos a - y\sin a, \tag{2.18}$$

$$y^* = x\sin a + y\cos a \tag{2.19}$$

generated by the infinitesimal operator

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{2.20}$$

If we use the equation (2.16), we seek

$$y_1(x, y) = k$$

as the general solution of

$$\frac{dy}{dx} = -\frac{x}{y}.$$

So,

$$y_1 = \sqrt{y + y^2} = \theta.$$

Now we are going to use formula (2.17) and solve the following equation:

$$\frac{dy_2}{dy} = \frac{1}{x} = \frac{1}{\sqrt{y_1^2 - y^2}}$$

A particular solution of equation gives us the second canonical coordinate as:

$$y_2 = \gamma = \arcsin\frac{y}{y_1}.$$

Therefore, we obtain canonical coordinates of rotations group (2.18), (2.19) as polar

coordinates

$$(y_1, y_2) = (\theta, \gamma) = (\sqrt{y + y^2},\ \arcsin \frac{y}{y_1}). \tag{2.21}$$

These coordinates are written with respect to the rotations group (2.18), (2.19) in the normal form

$$\theta^* = \theta,$$

$$\gamma^* = \gamma + a.$$

2.1.6 Lie Point Transformations and Prolongation Formulas

Vector spaces have extensions. We will now show how we pass to the extension of the infinitesimal operator by showing the extents of vector spaces. If we consider the variables as independent and dependent, we will divide the Lie group we are working with into two types.

Definition 2.4. The transformations of the Lie symmetry group we are working with are

$$x^* = X(x, y; a), \tag{2.22}$$

$$y^* = Y(x, y; a), \tag{2.23}$$

while $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y^1, y^2, \ldots, y^m) \in \mathbb{R}^m$.

is called a *one-parameter Lie point transformations group* in the space of sum of independent and dependent variables. Here, n is number of independent variables and m is number of dependent variables.

We note that the extension by partial derivatives with finite order. The infinitesimal operator X was acting on the dependent and independent variables. The prolongation of X also acts on derivatives of the dependent variables.

∂y: we have written here corresponds to the set of all partial derivatives of the dependent variable y with respect to the independent variable x.

$$\partial y = \left(\frac{\partial y^1}{\partial x_1}, \frac{\partial y^1}{\partial x_2}, \ldots, \frac{\partial y^1}{\partial x_n}, \frac{\partial y^2}{\partial x_1}, \frac{\partial y^2}{\partial x_2}, \ldots, \frac{\partial y^2}{\partial x_n}, \ldots, \frac{\partial y^m}{\partial x_1}, \frac{\partial y^m}{\partial x_2}, \ldots, \frac{\partial y^m}{\partial x_n} \right). \tag{2.24}$$

The number of elements of this set is nm. Mostly, for $p \geq 1$ the full set of pth-order partial derivatives of the dependent variable y with respect to the independent

variable x by $\partial^p y$ which consists of the elements

$$y^p_{i_1 i_2 \dots i_p} = \frac{\partial^p y^k}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_p}} \tag{2.25}$$

with $p = 1, 2, \dots, m$ and $i_j = 1, 2, \dots, n,\ j = 1, 2, \dots, p$ is written down.

Evidently, to alter partial derivatives of the dependent variables, we have to know prolongations of transformations in a one-parameter Lie group (2.22), (2.23) on the space of dependent variables and independent variables (x, y) to the space of dependent variables, independent variables and partial derivatives of dependent variables. In consequence, we prolong the infinitesimal operator of Lie group (2.22), (2.23) to infinitesimal operators in the space with derivatives.

The infinitesimal generator

$$prX = \xi^i \frac{\partial}{\partial x_i} + \eta^p \frac{\partial}{\partial y^k} + \tau^p_i \frac{\partial}{\partial y^p_i}, \tag{2.26}$$

where $i = 1, \dots, n$, $p = 1, \dots, m$ and τ^p_i are some functions of x_i, y^p and y^p_i is considered in order to see prolongation of a one-parameter transformation group to the space with first order derivatives.

The following cases tell us that under the invariant criterion, the geometric meanings of the derivatives are preserved as well as the definitions (2.22), (2.23):

$$d^p y = y^p_i dx^i,\ p = 1, 2, \dots, m. \tag{2.27}$$

The the total derivative operator with respect to the variable x^i:

$$D_i = \frac{\partial}{\partial x_i} + y^p_i \frac{\partial}{\partial y^p},\ i = 1, \dots, n, \tag{2.28}$$

Due to the invariant condition of differential equations, the coordinates (infinitesmals) of the infinitesimal operator, which we wrote with the prolongation form(2.26), must satisfy the following *prolongation formulas*::

$$\tau^p_i = D_i(\eta^p) - y^p_j D_i(\xi^j),\ i = 1, \dots, n,\ p = 1, \dots, m \tag{2.29}$$

Similarly, these formulas are obtained for equations with higher order derivatives. We

also obtain the second-order prolongation formulas as follows:

$$\tau^p_{ji} = D_i(\tau^p_j) - y^p_{sj} D_i(\xi^s),\ i,j = 1,\dots,n,\ p = 1,\dots,m. \tag{2.30}$$

Likewise, the same process can be applied for third or higher order derivatives. With prX we represent the infinitesimal operator with any order derivative

$$prX = \xi^i \frac{\partial}{\partial x_i} + \eta^p \frac{\partial}{\partial y^p} + \tau^p_i \frac{\partial}{\partial y^p_i} + \tau^p_{ij} \frac{\partial}{\partial y^p_{ij}} + \cdots . \tag{2.31}$$

From the prolongation formulas (2.29) and (2.30) it is clearly seen that the prolonged operator is linear and homogeneous. Although the main equation we are going to solve does not have to be linear, the equations are linear and homogeneous when we use prolongations.

2.1.7 Multiparameter Lie Transformation Groups and Lie Algebras

Up to the present, we have only defined and given examples of one-parameter Lie group transformations. But we seek that there are multi-parameter Lie transform groups with the help of dimensional analysis in transformations that we can say as scaling as an example. Just like in the one-parameter Lie group transformations, the fundamental element of the multiparameter Lie group transformations will be the infinitesimal operators. We can demonstrate every infinitesimal operator(generator) an exponential form which corresponds to a one-parameter Lie transformations group. Since this group, which we wrote in the mentioned form, will be a subgroup of the multi-parameter Lie group, we can consider the theory of infinitesimal operators with the same mentality as the theory of the multi-parameter Lie group. We can obtain a k-parameter Lie transformations group with infinitesimal operators of a one-parameter Lie group. We're going to show you how we got it by processes. Some operations will be defined between the infinitesimal operators of the one parameter Lie group.

We assume that

$$x^* = X(x;a) \tag{2.32}$$

is a *k-parameter Lie point transformations group* with variables $x = (x_1, x_2, \dots, x_n)$, parameters $a = (a_1, a_2, \dots, a_k)$. The composition operation for parameters represented by

$$\Gamma(a,b) = (\varphi_1(a,b), \varphi_2(a,b), \dots, \varphi_r(a,b)). \tag{2.33}$$

We assume that the group axioms are also satisfied. So the φ operation provides group axioms. For example, the identity element property for $a = 0$. We can write

$a_1 = a_2 = \cdots = a_k = 0$. Also, $\Gamma(a, b)$ is a differentiable function in its domain.

For multi-parameter transformation groups, there is an infinitesimal operator for each parameter. A group with k parameters can make us find at most k infinitesimal operators. The set of these operators is very important. This set forms the *Lie algebra*. Lie algebra is a k-dimensional vector space. We denote with L_k this vector space. In Lie algebra, the *commutator* operation is defined.

Definition 2.5. For an k-parameter Lie transformations group (2.32) with infinitesimal operators $X_r,\ r = 1, 2, \ldots, k$ and $X_s,\ s = 1, 2, \ldots, k$

$$[X_r, X_s] = X_r X_s - X_s X_r \tag{2.34}$$

is the *commutator (Lie bracket)* of X_r and X_s.

The commutator operation provides the following properties

- (*Antisymmetric or skew-symmetric*)

$$[X_r, X_s] = -[X_s, X_r],$$

- (*Bilinear*)

$$[\alpha X_r + \beta X_s, X_t] = \alpha[X_r, X_t] + \beta[X_s, X_t], (\alpha and \beta are constants.)$$

$$[X_r, \alpha X_s + \beta X_t] = \alpha[X_r, X_s] + \beta[X_s, X_t],$$

- (*Jacobi identity*)

$$[X_r, [X_s, X_t]] + [X_s, [X_t, X_r]] + [X_t, [X_s, X_r]] = 0.$$

Definition 2.6. If the commutator $[X_1, X_2] \in L_k$ when $X_1, X_2 \in L_k$. The vector space of infinitesimal operators of multiparameter Lie group L_k is called a *Lie algebra*.

Definition 2.7. Where U is a subset of L and $[X_r, X_s] \in U$ when $X_r, X_s \in U$ we say U is an *subalgebra* of a Lie algebra L. Additionally for all $X_1 \in U$, $X_2 \in L$, a subalgebra $U \subset L$ is called an *ideal* or *normal subalgebra* of L if $[X_1, X_2] \in U$.

Definition 2.8. If there is a chain of subalgebras, a Lie algebra L_k is called an k-dimensional *solvable Lie algebra*

$$\{0\} = L_0 \subset L_1 \subset \ldots \subset L_k$$

in such a way that $\dim(L_k) = k$ and L_{k-1} is a normal subalgebra of L_r for each m.

Lie algebras are subjects that help us a lot in solving higher order ODEs. We will seek how to apply Lie symmetry analysis to ODEs.

2.2 Application of Lie Group Analysis to Ordinary Differential Equations

Integration and Lie symmetry analysis are important methods in ordinary differential equations. A pth-order ODE will be geometrically corresponds to a surface in the $(p+2)$–dimensional space having elements. These elements are the independent variable, the dependent variable and derivatives of the dependent variable with order p. For that reason, we can think the solution set of the ODE as specific curves on this surface. Internally, when searching for a solution, the first integral describes a value in each solution curve. And a symmetry describes a mapping which moves each solution curve into solution curves. To be more precise, a transformation group with symmetry in Lie group analysis is a one-parameter transformation group that acts on the elements of $(p+2)$ dimensional space and maps a solution to an equivalent solution. A first integral is a quadrature. We describe it by a function of the independent variable, dependent variable and derivatives of dependent variable. The function' s order is $p-1$ and it is a constant for each solution.

Lie express that if the ordinary differential equation we are trying to solve admits a one-parameter Lie group of point transformation, we can reduce the order of this equation.We obtain the solution of our main equation from the solution of a reduced equation. We can reduce the order of the equation by k splitting the equation to an $(p-k)$th-order equation and k quadratures for a pth-order ODE if the corresponding symmetry group is an k–parameter solvable group of point transformations. But the order of reduced equation doesn't become $p-k$ according to the original independent and dependent variables. In other techniques of this method, like *integrating factor*, the order of the reduced equation is $p-k$ with respect to the original independent and dependent variables.

When it comes to first-order ODEs, the reduction by symmetries results in the quadrature of the equation. This allows us to find the first integral and the integration factor. The first integral we solve creates a quadrature. When we solve the first integral for a pth order ODE, the new order becomes $p-1$. So the order of the equation is decreased by 1. The order reduction and integration factor play a complementary role in the application of Lis symmetries to ODEs. There are also reduction methods other than the integration approach we mentioned. The order of an ODE can also be reduced by determining the differential invariants in the extended space or by using canonical

variables. As long as an ODE accepts the Lie transform group, the solution can be continued in many ways. We define these solutions by the curves which are invariant under the corresponding Lie transformations group and called invariant solutions. For first order ODEs invariant solutions could be computed in an algebraic way. We compute the invariant solutions of higher-order ODEs in an algebraic way or via the solution of the first-order ODE for the invariant curves of the group. In this section, we describe how Lie symmetries are applied to ODEs [7, 10].

We seek a one-parameter Lie group of point transformations

$$x^* = X(x, y; a), \;\; y^* = Y(x, y; a) \tag{2.35}$$

In addition the infinitesimal operator of this group is:

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2.36}$$

If the shape of an ODE is preserved with respect to transformations of variables (2.35), the equation

$$y' = \frac{dy}{dx} = f(x, y) \tag{2.37}$$

admits a one-parameter Lie group of point transformations (2.35) is called *point symmetry*.

That is, when we change the variables as below, the operation under the same function is provided. We can write for f :

$$\frac{dy^*}{dx^*} = f(x^*, y^*) \tag{2.38}$$

The same definitions apply to higher order equations.

We call an *infinitesimal symmetry* of the equation infinitesimal operator (2.36) of a Lie group admitted by a differential equation. Invariance condition of ODE (2.37) under a Lie group generated by operator (2.36) is displayed as:

$$X(y' - f(x, y)) = 0. \tag{2.39}$$

Consider the first order ODEs

$$\frac{dy}{dx} = f(x), \;\; \frac{dy}{dx} = g(y).$$

If we look at the solution curves of the first equation $y = u(x, c_1)$, they do not depend on y, since the equation is $dy/dx = f(x)$. Since the solution does not depend on y, we can do a shift on the y-axis. For example this leads us to a curve like $(x, y) \mapsto (x, y+c_1)$. Likewise, the solution from the second equation does not depend on x. So we can apply a shift operation on the x-axis. We can demonstrate x-direction into any other by mapping $(x, y) \mapsto (x + c_2, y)$.

Here are two methods to acquire the general solution of ODE (2.37) using the infinitesimals of an admitted point symmetry (2.35). These are the method of canonical coordinates and using an integrating factor.

2.2.1 The Method of Canonical Coordinates

Theorem 2.4. *A first-order ODE*

$$\frac{dy}{dx} = f(x, y) \tag{2.40}$$

admitting a Lie group as translations about its dependent variable is seperable.

Proof. Let $T : (x, y) \to (x^*, y^*) = (x, y + a)$ be a symmetry of equation (2.40) with $a \in \mathbb{R}$. Since

$$f(x, y + a) = f(x^*, y^*) = \frac{dy^*}{dx^*} = \frac{dy^*(x, y)}{dx^*(x, y)} = \frac{y^*_x + y' y^*_y}{x^*_x + y' x^*_y} = \frac{dy}{dx} = f(x, y),$$

the function f does not depend on y. So we have separable form of differential equation

$$\frac{dy}{dx} = f(x).$$

■

So Theorem 2.4 tells us , we can obtain the solution of the equation easily if a Lie group of a differential equation of form (2.40) is equivalent to a translational symmetry. Translational symmetry is transformed to the group translations about the dependent variable.

We can solve a linear or nonlinear first-order ODE

$$y' = \frac{dy}{dx} = f(x, y) \tag{2.41}$$

with a symmetry group generated by the operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{2.42}$$

by applying the following items in order.

- First we will solve the equations and get the canonical variables r,s. When the infinitesimal operator (2.42) is applied to variables, the following equations should be provided:

$$\begin{aligned} Xr &= 0, \\ Xs &= 1, \end{aligned} \tag{2.43}$$

- Now we are going to substitute the canonical variables r and s in equation (2.41). Here s becomes the new dependent variable and r becomes the new independent variable. And so we can easily calculate the integral of equation (2.41).

$$\frac{ds}{dr} = g(s) \quad \text{or} \quad \frac{ds}{dr} = h(r). \tag{2.44}$$

- We integrate equation (2.44). Then, substitute the statements $r = r(x,y)$ and $s = s(x,y)$ in the solution $s = \varphi(r,C)$, so the solution of equation (2.41) is gotten.

Example. [11] Consider the in the form of homogeneous differential equation

$$y' = \frac{y}{x} + (\frac{y}{x})^3. \tag{2.45}$$

In this example, we are going to obtain the solution of equation (2.45). And we will use the canonical variables. The equation (2.45) admits the group of scaling transformation.

$$x^* = e^a x, \;\; y^* = e^a y$$

with infinitesimal operator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{2.46}$$

where

$$\xi(x,y) = \left.\frac{\partial X(x,y;a)}{\partial a}\right|_{a=0} = \left.\frac{\partial e^a x}{\partial a}\right|_{a=0} = e^a x|_{a=0} = x,$$

and

$$\eta(x,y) = \left.\frac{\partial Y(x,y;a)}{\partial a}\right|_{a=0} = \left.\frac{\partial e^a y}{\partial a}\right|_{a=0} = e^a y|_{a=0} = y. \tag{2.47}$$

Actually

$$\frac{dy^*}{dx^*} = \frac{d(e^a y)}{d(e^a x)} = \frac{dy}{dx}. \tag{2.48}$$

Thus, $x^* = e^a x,\ \ y^* = e^a y$ leaves equation (2.45) invariant under a one-parameter Lie transformation group.

- We solve the first-order linear PDEs (2.43) that defines canonical variables r, s for (2.46), we get

$$r = \frac{y}{x},\ \ s = \ln x.$$

- After we substitute the canonical coordinates r, s in equation (2.45) we get:

$$y' = \frac{dy}{dx} = \frac{d(xr)}{dx} = r + x\frac{dr}{dx} = r + x\frac{dr}{ds}\frac{ds}{dx} = r + xr'\frac{1}{x} = r + r'.$$

  Then equation (2.45) becomes

$$\frac{ds}{dr} = \frac{1}{r^3}. \tag{2.49}$$

- Integration of equation (2.49) with canonical variables yields

$$s = -\frac{1}{2r^2} + C.$$

  Finally, we subsitute

$$r = \frac{y}{x}$$

  and

$$s = \ln x$$

2.2.2 Lie's Integrating Factor

Let' s consider a first-order ODE formed by

$$M(x, y)dx + N(x, y)dy = 0. \tag{2.50}$$

We assume the equation (2.50) admits a point transformations group with the infinitesimal operator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

and $\xi M + \eta N \neq 0$. We say the function

$$\mu = \frac{1}{\xi M + \eta N} \tag{2.51}$$

which is called *Lie's integrating factor* for the equation (2.50).

Example. [11] The Riccati-type of differential equation

$$y' = \frac{2}{x^2} - y^2, \ (x \neq 0) \tag{2.52}$$

admits the group of scaling transformations

$$x^* = e^a x, \ y^* = e^{-a} y, \tag{2.53}$$

generated by the operator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.54}$$

We can write the equation (2.52) in the form of equation (2.50) as

$$dy + \left(y^2 - \frac{2}{x^2}\right)dx = 0. \tag{2.55}$$

by substituting

$$\xi = x, \ \eta = -y, \ M = y^2 - \frac{2}{x^2}, \ N = 1,$$

into (2.51) gives the integrating factor

$$\mu = \frac{x}{x^2y^2 - xy - 2}.$$

We multiply equation (2.55) with the integrating factor. It helps us to find the solution of equation (2.52), Then we obtain the exact differential equation [9]:

$$\frac{x}{x^2y^2 - xy - 2}dy + \frac{1}{x^2y^2 - xy - 2}\frac{x^2y^2 - 2}{x}dx = 0. \tag{2.56}$$

Now, we will solve this equation with usual methods.

$$\frac{x^2y^2 - 2}{x} = y + \frac{x^2y^2 - xy - 2}{x}.$$

We write again equation (2.56) as

$$\frac{d(xy)}{x^2y^2 - xy - 2} + \frac{dx}{x} = 0. \tag{2.57}$$

We will use the transform of $v = xy$ for the first term of this equation.

$$\frac{d(xy)}{x^2y^2 - xy - 2} = \frac{dv}{v^2 - v - 2} = \frac{1}{3}\left(\frac{1}{v-2} - \frac{1}{v+1}\right)dv. \tag{2.58}$$

After we integrate this term results with

$$\int \frac{1}{v^2-v-2}dv = \frac{1}{3}\ln\left(\frac{v-2}{v+1}\right). \tag{2.59}$$

Returning equation (2.57) with first variables gives

$$d\left(\frac{1}{3}\ln\left(\frac{xy-2}{xy+1}\right)+\ln x\right)=0. \tag{2.60}$$

Finally, we integrate this equation. And we obtain the solution of equation (2.52) as

$$y=\frac{2x^3+c}{x(x^3-c)}. \tag{2.61}$$

C is an arbitrary constant. The functions $y=2/x$ and $y=-1/x$ provide differential equation (2.52). They are also solutions of the equation and we call them *invariant solutions*.

2.2.3 Invariant Solutions

A Lie transformations group admitted by an ODE is very important because it allows us to seek solutions. The symmetry transformations map a solution of the equation into equivalent solution. Thus, the transformations of the group replace the integral curves with each other by leaving some of the integral curves the same. We call the integral curves invariant solutions.

Invariant solutions shows a simple way to obtain general solutions to differential equations. This method is used for first order ODEs with two given infinitesimal operators.

Example. (*Invariant solutions of an ODE*)

We consider the equation

$$y'=\frac{y}{x}+\frac{y^2}{x^3} \tag{2.62}$$

and we know two infinitesimal symmetries

$$X_1=x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y},\ X_2=x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}.$$

We found from the invariance criteria $X_2(I)=0$, the only independent invariant y/x^2. Then, we obtain invariant solution by taking $p=y/x^2$ or $y=px^2$ with an arbitrary constant $p\neq 0$. In the circumstances , the equation (2.62) reduces to

$$y' - \frac{y}{x} - \frac{y^2}{x^3} = 2px - px - p^2x = p(1-p)x = 0.$$

In this case $p = 1$, the invariant solution is

$$y = x^2. \tag{2.63}$$

Now, under the projective transformation

$$x^* = \frac{x}{1-ax},\ y^* = \frac{y}{1-ax} \tag{2.64}$$

generated by X_1. Then we write the invariant solution (2.63) in the form of new variables

$$y^* = (x^*)^2,$$

and substitute the expressions (2.64) for x^* and y^*, the equation transforms

$$\frac{y}{1-ax} = \frac{x^2}{(1-ax)^2}.$$

In the last step, we denote the parameter a by C and obtain general solution of equation (2.62) as

$$y = \frac{x^2}{1-Cx}. \tag{2.65}$$

2.2.4 Applications of Lie Symmetries to Higher Order Ordinary Differential Equations

If there exists a multiparameter Lie transformations group admitted by the equation, we can reduce the order of a higher order ODE. We will do that step-by-step. In addition, if an ODE of order n has an k-parameter group we can reduce the order of the differential equation to $n-k$. It should also be noted that the Lie algebra corresponding to the group must be solvable when applying the reduction operation.

Consider an ODE of the form with nth order

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ n \geq 2. \tag{2.66}$$

If X is the infinitesimal operator of a one-parameter group of (2.66) and

$r(x,y)$, $s(x,y)$ are the canonical variables, we have

$$X(r,s) = \frac{\partial}{\partial s}.$$

After we write (2.66) in terms of canonical variables gives

$$s^{(n)} = \phi(r,s,s',...,s^{(n-1)}), \quad s^{(k)} = \frac{d^k s}{dr^k} \tag{2.67}$$

for some function Φ. Since (2.67) is invariant under the group of translations in s-direction, we have that

$$\Phi_s = 0.$$

Therefore, (2.67) takes the form

$$s^{(n)} = \Phi(r,,s',...,s^{(n-1)}).$$

Thus, for $v = ds/dr$ writing (2.66) in terms of canonical variables reduces the order of (2.66) to the following ODE of order $n-1$,

$$v^{(n-1)} = \Phi(r,v,...,v^{(n-2)}), \quad v^{(p)} = \frac{d^{p+1}s}{dr^{p+1}}.$$

Example. [12] Let us explain the example given by Vladimir Dorodnitsyn in his book for a second-order, non-linear ODE.

$$v'' = \frac{1}{v^3}, \tag{2.68}$$

We write the infinitesimal of the operator:

$$X = \xi(t,v)\frac{\partial}{\partial x} + \eta(t,v)\frac{\partial}{\partial v}. \tag{2.69}$$

We must write the prolongation of X to the first and second dervatives:

$$prX = \xi(t,v)\frac{\partial}{\partial x} + \eta(t,v)\frac{\partial}{\partial v} + \tau_1\frac{\partial}{\partial v'} + \tau_2\frac{\partial}{\partial v''}, \tag{2.70}$$

$$prX(v'' - \frac{1}{v^3}) = 0$$

must be satisfied for invariance criteria. After we apply the theorem in which the invariant condition is satisfied to the equation, we get the following equation.

$$\tau_2 = -\frac{3\eta}{v^4} \tag{2.71}$$

Now we will use the formulas with total derivatives.

$$\tau_1 = D(\eta) - v' D(\xi), \tag{2.72}$$

$$\tau_2 = D(\tau_1) - v'' D(\xi) = D^2(\eta) - 2v'' D(\xi) - v' D^2(\xi), \tag{2.73}$$

$$D = \frac{\partial}{\partial t} + v' \frac{\partial}{\partial v} + v'' \frac{\partial}{\partial v'} + v''' \frac{\partial}{\partial v''} + \cdots . \tag{2.74}$$

We equate the τ_2 we found in the equation to each other. This will help us obtain the determining equation.

$$D^2(\eta) - 2v'' D(\xi) - v' D^2(\xi) = -\frac{3\eta}{v^4},$$

$$\begin{aligned} &\eta_{tt} + 2\eta_{tv} v' + \eta_{vv}(v')^2 + \eta_v v'' - 2v''(\xi_t + \xi_v v') \\ &- v'(\xi_{tt} + 2\xi_{tv} v' + \xi_{vv}(v')^2 + \xi_v v'') + \frac{3}{v^4}\eta = 0. \end{aligned} \tag{2.75}$$

After substituting the equation provided by the equation (2.68), we obtained the determining equation:

$$\begin{aligned} &\eta_{tt} + 2\eta_{tv} y' + \eta_{vv}(v')^2 + \eta_v \frac{1}{v^3} - 2\frac{1}{v^3}(\xi_t + \xi_v v') \\ &- v'(\xi_{tt} + 2\xi_{tv} y' + \xi_{vv}(v')^2 + \xi_v \frac{1}{v^3}) + \frac{3}{v^4}\eta = 0. \end{aligned} \tag{2.76}$$

Each solution(non-zero) of the determining equation will let us to find the infinitesimals of the operator. We see the powers of the derivatives of v in the determining equation. We write the equation as the polynomial of v' and try to find a solution by setting the coefficients to zero.

$$\begin{aligned} \xi_{vv} = 0,\ \ \eta_{vv} - 2\xi_{tv} = 0, \\ 2\eta_{tv} - 3\frac{\xi_v}{v^3} - \xi_{tt} = 0, \\ \eta_{tt} + (\eta_v - 2\xi_t)\frac{1}{v^3} + \frac{3\eta}{v^4} = 0. \end{aligned} \tag{2.77}$$

If we solve the integrals of the first two partial differential equations in (2.68) and substitute the solutions in the last two equations, we can write the infinitesimals as follows.

$$\xi = f(v)t^2 + 2g(v)t + h(v),\ \ \eta = v(f(v)t + g(v)).$$

When we arrange the u coefficient equation and write it as a polynomial according to

the powers of u. We obtain the following general solution forms.

$$\xi(t,v) = \alpha t^2 + 2\beta t + \theta,\ \ \eta(t,v) = (\alpha t + \beta)v,$$

Here α, β, θ are arbitrary constants. We have obtained the 3D solution space of the operators. When we go by giving 1 of the arbitrary constants and 0 to the others, we get a base of the symmetry operator.

$$X_1 = \frac{\partial}{\partial t},\ X_2 = 2\frac{\partial}{\partial t} + v\frac{\partial}{\partial v},\ X_3 = t^2\frac{\partial}{\partial t} + tv\frac{\partial}{\partial v}. \tag{2.78}$$

We also demonstrate the following the point transformations using Lie equations. We have also shown that our solutions are invariant.

$$\begin{aligned} t^* = t + a,\ \ v^* = v;\\ t^* = e^{2a}t,\ \ v^* = e^a v;\\ t^* = \frac{t}{1-at},\ \ v^* = \frac{v}{1-at}. \end{aligned} \tag{2.79}$$

Here are the prolongations of transformations (2.79) to the first and second derivatives as

$$\begin{aligned} t^* = t + a,\ \ v^* = v,\ \ (v')^* = v',\ \ (v'')^* = v'';\\ t^* = e^{2a}t,\ \ v^* = e^a v,\ \ (v')^* = e^{-a}v',\ \ (v'')^* = e^{-3a}v'';\\ t^* = \frac{t}{1-at},\ \ v^* = \frac{v}{1-at},\\ (v')^* = av + (1-at)v',\ \ (v'')^* = (1-at)^3 v''. \end{aligned} \tag{2.80}$$

When we substitute these transformations in (2.68) equation, we see that the solution does not change. In short, we have obtained an equivalent solution.

$$(v'')^* = \frac{1}{(v^*)^3}$$

at the point $(t^*, v^*, (v')^*, (v'')^*)$. Consequently, the symmetry group leaves invariant the equation, but moves the point by the transform $(t, v, v', v'') \to (t^*, v^*, (v')^*, (v'')^*)$.

2.3 Application of Lie Group Analysis to Partial Differential Equations

While applying the symmetry method to partial differential equations, boundary conditions and the invariance of the equation can cause a reduction in the number of independent variables. These boundary value conditions are under a one-parameter Lie group transformation. The invariants of the group is going to be the new variables.

The infinitesimal operator of a Lie point transformations group admitted by a given PDE is determined by an algorithm with respect to the infinitesimal criterion for the invariance of a PDE. Thanks to the invariance of a PDE under the group of Lie point transformations, we can obtain special solutions. We call these particular solutions invariant solutions or similarity solutions.

Similarity solutions are invariant under a subgroup of the Lie group admitted by the PDE. At the same time thet provides solving PDEs with fewer independent variables than in the given PDE. Invariant solutions could be acquired for BVPs. In this instance, the solutions are invariant under a subgroup of a Lie group admitted by the given PDE that leaves invariant the corresponding boundary curves and boundary conditions [9].

We can obtain invariant solutions of a scalar PDE or a system of PDE using an admitted point transformations group. We will use two methods:

1. *The Method of Invariant Form*. We solve the characteristic equations resulting from the invariance infinitesimal criteria and obtain the invariant form. Then solution of the given PDE gives the invariant solution.

2. *The Direct Substitution Method*. We select an independent variable as a parameter. We substitute invariance conditions and also necessary differential results into the given PDE. Thus, we reduce all derivatives with respect to the parametric independent variable are annihilated and one independent variable in the original PDE. We substitute solution of the (reduced) PDE into either invariance conditions. We can do same thing to the given PDE gives the invariant solutions. In this method, invariant solutions are determined without solving characteristic equations [13].

   These two methods could be extended for obtaining invariant solutions (of PDEs) from an admitted multiparameter point transformations group.

If we interpret Lie group of point transformations of a PDE geometrically, it is an movement in its solution set. Similar to the ODEs, a Lie group of point transformations admitted by a PDE geometrically describes an integral curve of a vector field. This vector field is tangent to the surface defined by the given PDE. Point symmetries of PDEs also preserve the meaning of the derivatives in the prolonged space that includes coordinates given by the independent variables, dependent variables and all their partial derivatives up to a fixed finite order. The overlapping of all point symmetries admitted by a PDE creates a Lie group with an algebraic structure. We represent the Lie algebra of the group with the commutator operation we were familiar with. This

commutator operation is defined by operations of infinitesimal operators which the point symmetries admitted by the given PDE.

Now, we are going to express constructing the solutions of PDEs with the main idea of invariance under Lie point transformations groups. These are presented in [8].

2.3.1 Invariance Criteria for Partial Differential Equations

We will define symmetry groups for PDEs in the same way as for ODEs.

Definition 2.9. We consider a pth-order PDE

$$F(x, y, \partial y, \partial^2 y, \ldots, \partial^p y) = 0, \tag{2.81}$$

where $x = (x_1, x_2, \ldots, x_n)$ expresses the n independent variables and y expresses the dependent variable. $\partial^m y$ expresses the m-th order partial derivatives of y with respect to x with $\partial^l y / \partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_m} = y_{i_1 i_2 \ldots i_j}$, $i_m = 1, 2, \ldots, n$, for $m = 1, 2, \ldots, p$.

The PDE is invariant under one-parameter Lie transformations group

$$\begin{aligned} x^* &= X(x, y; a), \\ y^* &= Y(x, y; a), \end{aligned} \tag{2.82}$$

if and only if its p-th extension, we define it by

$$\begin{aligned} x^* &= X(x, y; a), \\ y^* &= Y(x, y; a), \\ \partial y^* &= \partial Y(x, y, \partial y; a), \\ &\vdots \\ \partial^p y^* &= \partial^p Y(x, y, \partial y, ..., \partial^p y; a), \end{aligned}$$

leaves invariant the vector field (2.81). We denote this case by transformations group (2.82) is a point symmetry admitted by PDE (2.81).

If (2.81) admits (2.82), a one-parameter Lie group of point transformations (2.82) leaves invariant the set of all solutions of PDE (2.81).

Theorem 2.5 (Infinitesimal Operator for the Invariance of a PDE)**.** *The infinitesimal operator of Lie point transformations group (2.82)*

$$X = \xi_i(x, y) \frac{\partial}{\partial x_i} + \eta(x, y) \frac{\partial}{\partial y} \tag{2.83}$$

and p-th extended infinitesimal generator of (2.83), where $\eta_i^{(1)}$ *and* $\eta_{i_1 i_2 \ldots i_j}^{(j)}$, $i_j = 1, 2, \ldots, n$ *for* $j = 1, 2, \ldots, k$ *are the extended infinitesimals*

$$\begin{aligned} X^{(p)} =& \xi_i(x,y)\frac{\partial}{\partial x_i} + \eta(x,y)\frac{\partial}{\partial y} + \eta_i^{(1)}(x,y,\partial y)\frac{\partial}{\partial y_i} + \cdots \\ &+ \eta_{i_1 i_2 \ldots i_p}^{(p)}(x,y,\partial y,\partial^2 y,\ldots,\partial^p y)\frac{\partial}{\partial y_{i_1 i_2 \ldots i_p}} \end{aligned} \tag{2.84}$$

in terms of $\xi(x,y) = (\xi_1(x,y), \xi_2(x,y), \ldots, \xi_n(x,y))$, $\eta(x,y)$.

In addition, one-parameter Lie point transformations group (2.82) is admitted by PDE (2.81). So, (2.82) is a point symmetry of PDE (2.81) if and only if

$$X^{(p)}F(x,y,\partial y,\partial^2 y,\ldots,\partial^p y) = 0 \quad when \quad F(x,y,\partial y,\partial^2 y,\ldots,\partial^p y) = 0. \tag{2.85}$$

2.3.2 Invariant Solutions for Partial Differential Equations

Definition 2.10. $y = \phi(x)$ is an *invariant solution* of PDE (2.81) obtained from its admitted point symmetry generated by operator (2.83) if and only if:

1. $y = \phi(x)$ is an invariant surface of (2.83);
2. $y = \phi(x)$ is a solution of (2.81).

As a result of this definition, we can write the following results.

For a solution of PDE (2.81) to be an invariant solution under point symmetry generated by operator (2.83), the necessary and sufficient conditions are

- $X(y - \phi(x))$ when $y = \phi(x)$. This results with the equation

$$\xi_i(x,\phi(x))\frac{\partial\phi(x)}{\partial x_i} = \eta(x,\phi(x)); \tag{2.86}$$

- $F(x,y,\partial y,\partial^2 y,\ldots,\partial^k y) = 0$ when $y = \phi(x)$, i.e.,

$$F(x,\phi(x),\partial\phi(x),\partial^2\phi(x),\ldots,\partial^k\phi(x)) = 0. \tag{2.87}$$

We can obtain invariant solutions of PDEs obtained in two ways:

(1) Invariant Form Method

We start with solving first-order PDE (2.86).

$$\frac{dx_1}{\xi_1(x,y)} = \frac{dx_2}{\xi_2(x,y)} = \cdots = \frac{dx_n}{\xi_n(x,y)} = \frac{dy}{\eta(x,y)}. \tag{2.88}$$

are the corresponding characteristic equations for PDE with $y = \phi(x)$.

We have got n functionally independent constants by solving of the system of n first-order ODEs (2.88). These are $c_1(x,y), c_2(x,y), \ldots, c_{n-1}(x,y)$. We obtain the general solution $y = \phi(x)$ of PDE (2.86) is given in the invariant form

$$t(x,y) = \Gamma(c_1(x,y), c_2(x,y), \ldots, c_{n-1}(x,y)), \tag{2.89}$$

where Γ is an arbitrary differentiable function of $c_1(x,y), c_2(x,y), \ldots, c_{n-1}(x,y)$. Since $c_1(x,y), c_2(x,y), \ldots, c_{n-1}(x,y), s(x,y)$ are n group invariants of point symmetry (2.83), they form n canonical variables for the Lie point transformations group (2.82). Let $c_n(x,y)$ be the $(n+1)$-th canonical variable satisfies the following equation.

$$Xc_n = 1. \tag{2.90}$$

After we substitute these canonical variables in PDE (2.81), we obtain a p-th order PDE with independent variables $c_1, c_2, \ldots, c_n$ and dependent variable t and this new PDE would admit the one-parameter Lie translations group

$$\begin{aligned} {c_i}^* &= c_i, \ \ i = 1, 2, \ldots, n-1, \\ {c_n}^* &= c_n + a, \\ t^* &= t. \end{aligned}$$

Hence, the variable c_n would not seen clearly in the new PDE. We obtain solutions implicitly in the form (2.89) would be solutions of the new PDE. These are invariant solutions of PDE (2.81). After all , such solutions are determined solving a reduced differential equation with $n-1$ independent variables $c_1, c_2, \ldots, c_{n-1}$ which are called similarity variables and a dependent variable t. The reduced differential equation is obtained rewriting PDE (2.81) in terms of invariant form (2.89).

Then we have reduced the given PDE is reduced to an ODE where $c_i = c_i(x)$, $i = 1, 2, \ldots, n-1$ and if $n = 2$.

(2) Direct Substitution Method

In this method, we do not need to solve invariance condition (2.86) or characteristic equations (2.88).

We seek the the equation form

$$\frac{\partial y}{\partial x_n} = \frac{\eta(x,y)}{\xi_n(x,y)} - \sum_{i=1}^{n-1} \frac{\xi_i(x,y)}{\xi_n(x,y)} \frac{\partial y}{\partial x_i}. \tag{2.91}$$

We will solve the equation which is demonstrated from first-order PDE (2.86) with hypothesis $\xi_n(x,y) \neq 0$. Then we substitute equation (2.91) and its differential relations directly into given PDE (2.81). For each element in (2.81) including derivatives of y with respect to x_n gives a decreased differential equation. The reduced equation has order at most p including the dependent variable y. Every solution of this reduced differential equation describes an invariant solution for PDE (2.81) with respect to the invariance under point symmetry (2.83). They provide invariance condition (2.91) and given PDE (2.81). If n=2 then the given PDE is reduced to an ODE. This method is easier than the Invariant Form Method. Because in here we do not have to deal with integration of the characteristic equations.

We consider a p-th order PDE ($p \geq 2$, $m \leq p$)

$$y_{i_1 i_2 \ldots i_m} = f(x, y, \partial y, \partial^2 y, \ldots, \partial^p y), \tag{2.92}$$

where $f(x, y, \partial y, \partial^2 y, \ldots, \partial^p y)$ is independent of $y_{i_1 i_2 \ldots i_m}$. PDE (2.92) admits the one-parameter Lie point transformations group. The group is generated by the operator

$$X = \xi_i(x,y) \frac{\partial}{\partial x_i} + \eta(x,y) \frac{\partial}{\partial y}. \tag{2.93}$$

We formulate p-th extension (2.84), if and only if $\xi(x,y)$ and $\eta(x,y)$ satisfy the determining equation

$$\eta^{(l)}_{i_1 i_2 \ldots i_l} = \xi_j \frac{\partial f}{\partial x_j} + \eta \frac{\partial f}{\partial y} + \eta^{(1)}_j \frac{\partial f}{\partial y_j} + \cdots + \eta^{(p)}_{j_1 j_2 \ldots j_p} \frac{\partial f}{\partial y_{j_1 j_2 \ldots j_p}}. \tag{2.94}$$

Also y satisfies (2.92).

2.3.3 Examples

We show some applications of Lie symmetries to PDEs in the following examples.

Example. [11]

The (linear) heat equation

$$u_t - u_{xx} = 0 \tag{2.95}$$

is invariant under the Galilean transformation.

$$t^* = t,\ x^* = x + 2at,\ u^* = ue^{-(ax+a^2t)} \tag{2.96}$$

The group is generated by the operator

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}. \tag{2.97}$$

Each solution

$$u = \psi(t, x)$$

of the heat equation can be mapped into a new solution by the group of transformations (2.96). We can write the heat equation in the form $u^*_{t^*} - u^*_{x^*x^*} = 0$. Because it is invariant under this group. After we use the new variables t^*, x^*, u^*, write the solution as

$$u^* = \psi(t^*, x^a st).$$

Then we substitute the expressions (2.96) for t^*, x^*, u^*, and solve the resulting equation

$$ue^{-(ax+a^2t)} = \psi(t, x + 2at)$$

for u, the new solution

$$u = e^{ax+a^2t}\psi(t, x + 2at) \tag{2.98}$$

with the parameter a. For instance, for the simple solution $u = x$, letting $\psi(t, x + 2at) = x + 2at$ in (2.98), we obtain the new solution as

$$u = (x + 2at)e^{ax+a^2t}.$$

Example. [11] (*The Burgers Equation*)

The Burgers equation

$$u_t = u_{xx} + uu_x \tag{2.99}$$

has five given independent symmetry operators

$$\begin{gathered} X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \\ X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\ X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}. \end{gathered} \tag{2.100}$$

We write the group of transformations which are generated by operator X_5

$$t^* = \frac{t}{1-at},\ x^* = \frac{x}{1-at},\ u^* = u(1-at) - ax. \tag{2.101}$$

We use these transformations. Any solution that we know $u = \Psi(t,x)$ of the Burgers equation is mapped to the following one parameter set of new solutions:

$$u = \frac{ax}{1-at} + \frac{1}{1-at}\Psi\left(\frac{t}{1-at}, \frac{x}{1-at}\right).$$

Because of the invariance of the equation (2.99) with respect to the symmetry group of translation in time ($x^* = x,\ t^* = t+a,\ u^* = u$) generated by operator X_1 gives the solution

$$u = \Psi(x)$$

which is an important solution in studies of physics. After we substitute this solution into Burgers equation we get

$$\Psi'' + \Psi\Psi' = 0. \tag{2.102}$$

We integrate (2.102) and we get

$$\Psi' + \frac{\Psi^2}{2} = C_1,$$

Then, we integrate again by setting $C_1 = 0,\ C_1 = v^2 > 0,\ C_1 = -w^2 < 0$ and obtain the solutions:

$$\begin{aligned} \Psi(x) &= \frac{2}{x+C}, \\ \Psi(x) &= v\tanh\left(C + \frac{v}{2}x\right), \\ \Psi(x) &= w\tan\left(C - \frac{w}{2}x\right) \end{aligned} \tag{2.103}$$

3
MAIN RESULTS

In this part of our thesis we analyzed the coupled system of the sine-Gordon equations. When we examine the academic publications, we see that different methods are applied for the solution of the sine-Gordon equations. This system of equations has received great attention because of its soliton solutions. We applied Lie symmetry analysis to the sine-Gordon equations. We will consider Lie group transformations that leave them invariant.

$$\begin{cases} u_{tt} - u_{xx} = \sin(u-v), \\ v_{tt} - v_{xx} = \sin(u-v) \end{cases} \tag{3.1}$$

(3.1) is a system of equations consisting of one parameter a. The Lie equations we explained in the previous section. We will work on Lie point transformations using the formula for the extension of the infinitesimal operator.

$$x^* = x^*(x,t,u,v;a)$$

$$t^* = t^*(x,t,u,v;a)$$

$$u^* = u^*(x,t,u,v;a)$$

$$v^* = v^*(x,t,u,v;a)$$

As above, a Lie transform group containing the independent variables x,t and the dependent variables u and v is considered.

We use the infinitesimal operator of the symmetry group in the form:

$$X = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} \tag{3.2}$$

We need the following prolonged form of X

$$\begin{aligned} prX = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} \\ + \tau_1 \frac{\partial}{\partial u_x} + \tau_2 \frac{\partial}{\partial u_t} + \tau_3 \frac{\partial}{\partial v_x} + \tau_4 \frac{\partial}{\partial v_t} \\ + \tau_5 \frac{\partial}{\partial u_{xx}} + \tau_6 \frac{\partial}{\partial u_{xt}} + \tau_7 \frac{\partial}{\partial u_{tt}} + \tau_8 \frac{\partial}{\partial v_{xx}} + \tau_9 \frac{\partial}{\partial v_{xt}} + \tau_{10} \frac{\partial}{\partial v_{tt}} \end{aligned} \tag{3.3}$$

So the one parameter Lie transform group which is corresponds to the sine-Gordon equations is:

$$\begin{aligned} x^* = e^{aX}(x) = x^*(x,t,u,v;a) + O(a^2) \\ t^* = e^{aX}(t) = t^*(x,t,u,v;a) + O(a^2) \\ u^* = e^{aX}(u) = u^*(x,t,u,v;a) + O(a^2) \\ v^* = e^{aX}(v) = v^*(x,t,u,v;a) + O(a^2) \end{aligned}$$

When the first prolongation of the infinitesimal generator belonging to this Lie group is applied to the differential equation. The result must be zero on the manifold defined by the differential equation. Thus this differential equation system will admit the given Lie transformation group. We apply prX to equations in (3.1).

So we have:

$$\tau_7 - \tau_5 - \eta^u \cos(u-v) + \eta^v \cos(u-v) = 0 \tag{3.4}$$

$$\tau_{10} - \tau_8 - \eta^u \cos(u-v) + \eta^v \cos(u-v) = 0 \tag{3.5}$$

We have to obtain the determining equation and we will find the nonzero solutions. These solutions give us the infinitesimals of the operator generating a one-parameter group.

We will now apply the prolongation formulas whose definitions we have explained in Chapter 2. These formulas will also give us expressions that are equal to $\tau_5, \tau_7, \tau_8, \tau_{10}$.

We write the formulas in which we use the total derivative. And in line with these formulas, we apply the total derivatives to the infinitesimals.

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + v_{xt} \frac{\partial}{\partial v_x} + v_{xxx} \frac{\partial}{\partial v_{xx}}$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_t} + u_{ttt} \frac{\partial}{\partial u_{tt}} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_t} + v_{ttt} \frac{\partial}{\partial v_{tt}}$$

$$\tau_1 = D_x(\eta^u) - u_x D_x(\xi^x) - u_t D_x(\xi^t)$$

$$\tau_1 = \eta^u_x + u_x \eta^u_u + v_x \eta^u_v - u_x(\xi^x_x + u_x \xi^x_u + v_x \xi^x_v) - u_t(\xi^t_x + u_x \xi^t_u + v_x \xi^t_v)$$

$$\tau_5 = D_x(\tau_1) - u_{tx} D_x(\xi^x) - u_{xx} D_x(\xi^t)$$

$$\tau_5 = \eta^u_{xx} + u_x \eta^u_{xu} + v_x \eta^u_{xv} + u_x \eta^u_{ux} + (u_x)^2 \eta^u_u + v_x u_x \eta^u_{uv}$$
$$+ v_x \eta^u_{vx} + (u_x v_x) \eta^u_{vu} + (v_x)^2 \eta^u_{vv} - u_x \xi^x_{xx} - (u_x)^2 \xi^x_{xu}$$
$$+ v_x u_x \xi^x_{xv} - (u_x)^2 \xi^x_{ux} - (u_x)^3 \xi^x_u - v_x (u_x)^2 \xi^x_{uv}$$
$$- (u_x v_x) \xi^x_{vx} - (u_x)^2 v_x \xi^x_{vu} - u_x (v_x)^2 \xi^x_{vv} - u_t \xi^t_{xx}$$
$$- (u_x u_t) \xi^t_{xu} - (u_t v_x) \xi^t_{xv} - (u_t u_x) \xi^t_{ux} - u_x^2 u_t \xi^t_u$$
$$- (u_t u_x v_x) \xi^t_{uv} - (u_t v_x) \xi^t_{vx} - (u_x u_t v_x) \xi^t_v - (v_x)^2 u_t \xi^t_{vv}$$
$$- u_{tx}(\xi^x_x + u_x \xi^x_u + v_x \xi^x_v) - u_{xx}(\xi^t_x + u_x \xi^t_u + v_x \xi^t_v) = 0$$

$$\tau_2 = D_t(\eta^u) - u_x D_t(\xi^x) - u_t D_t(\xi^t)$$

$$\tau_2 = \eta^u_t + u_t \eta^u_u + v_t \eta^u_v - u_x(\xi^x_t + u_t \xi^x_u + v_t \xi^x_v) - u_t(\xi^t_t + u_t \xi^t_u + v_t \xi^t_v)$$

$$\tau_7 = D_t(\tau_2) - u_{tx} D_t(\xi^x) - u_{tt} D_t(\xi^t)$$

$$\tau_7 = \eta^u_{tt} + u_t \eta^u_{tu} + v_t \eta^u_{tv} + u_t \eta^u_{ut} + (u_t)^2 \eta^u_u + v_t u_t \eta^u_{uv}$$
$$+ v_t \eta^u_{vt} + u_t v_t \eta^u_{vu} + (v_t)^2 \eta^u_{vv} - u_x \xi^x_{tt} - u_x u_t \xi^t_{tu}$$
$$+ v_t u_x \xi^x_{tv} - u_x u_t \xi^t_{ut} - u_x (u_t)^2 \xi^x_u - u_x u_t v_t \xi^x_{uv}$$
$$- u_x v_t \xi^x_{vt} - u_x v_t \xi^x_{vt} - u_x v_t u_t \xi^x_{vu} - u_x (v_t)^2 \xi^x_{vv}$$
$$- u_t \xi^t_{tt} - (u_t)^2 \xi^t_{tu} - u_t v_t \xi^t_{tv} - (u_t)^2 \xi^t_{ut} - (u_t)^3 \xi^t_u$$

$$-(u_t)^2 v_t \xi^t_{uv} - u_t v_t \xi^t_{vt} - (u_t)^2 v_t \xi^t_{vu} - (u_t)^2 v_t \xi^t_{vu}$$
$$-u_t (v_t)^2 \xi^t_{vv} - u_{tx}(\xi^x_t + u_t \xi^x_u + v_t \xi^x_v) - u_{tt}(\xi^t_t + u_t \xi^t_u + v_t \xi^t_v) = 0$$

If we substitute τ_7, τ_5 where we found at (3.4) we have:

$$\xi^x_v = 0, \xi^x_u = 0, \xi^x_x = \xi^x_t$$
$$\xi^t_u = 0, \xi^t_x = 0, \xi^t_v = 0, \xi^t_t = 0$$
$$\eta^u_{ut} = 0, \eta^u_u = 0, \eta^u_{vt} = 0, \eta^u_{uv} = 0, \eta^u_{vv} = 0, \eta^u_{vx} = 0$$
$$-\xi^x_{tt} - 2\eta^u_{xu} - \xi^x_{xx} = 0$$
$$\eta^u_{tt} - \eta^u_{xx} - \eta^u \cos(u - v) + \eta^v \cos(u - v) = 0$$

$$\tau_3 = D_x(\eta^v) - v_x D_x(\xi^x) - v_t D_x(\xi^t)$$
$$\tau_3 = \eta^v_x + u_x \eta^v_u + v_x \eta^v_v - v_x(\xi^x_x + u_x \xi^x_u + v_x \xi^x_v) - v_t(\xi^t_x + u_x \xi^t_u + v_x \xi^x_v)$$

$$\tau_8 = D_x(\tau_3) - v_{tx} D_x(\xi^x) - v_{xx} D_x(\xi^t)$$
$$\tau_8 = \eta^v_{xx} + u_x \eta^v_{xu} + v_x \eta^v_{xv} + u_x \eta^v_{ux} + (u_x)^2 \eta^v_u + v_x u_x \eta^v_{uv}$$
$$+v_x \eta^v_{ux} + u_x v_x \eta^v_u + (v_x)^2 \eta^v_{uv} - v_x \xi^x_{xx} - u_x v_x \xi^x_{xu}$$
$$-(v_x)^2 \xi^x_{xv} - (u_x v_x) \xi^x_{ux} - v_x (u_x)^2 \xi^x_u - u_x (v_x)^2 \xi^x_{uv}$$
$$-(v_x)^2 \xi^x_{vx} + u_x (v_x)^2 \xi^x_{vu} - (v_3)^3 \xi^x_v - v_t \xi^t_{xx} - u_x v_t \xi^t_{xu}$$
$$-v_x v_t \xi^t_{xv} - v_t u_x \xi^t_{ux} + v_t (u_x)^2 \xi^t_u - v_t u_x v_x \xi^t_{uv} - v_t v_x \xi^x_{vx}$$
$$-v_t v_x u_x \xi^x_{vu} - (v_x)^2 u_t \xi^x_{vv} - v_{tx}(\xi^x_x + u_x \xi^x_u + v_x \xi^x_v)$$
$$-v_{xx}(\xi^t_x + u_x \xi^t_u + v_x \xi^t_v) = 0$$

$$\tau_4 = D_t(\eta^v) - v_x D_t(\xi^x) - v_t D_t(\xi^t)$$
$$\tau_4 = \eta^v_t + u_t \eta^v_u + v_t \eta^v_v - v_x(\xi^x_t + u_t \xi^x_u + v_t \xi^x_v) - v_t(\xi^t_t + u_t \xi^x_u + v_t \xi^t_v)$$

$$\tau_{10} = D_t(\tau_4) - v_{tx} D_t(\xi^x) - v_{tt} D_t(\xi^t)$$

$$\tau_{10} = \eta^v_{tt} + u_t\eta^v_{tu} + v_t\eta^v_{tv} + u_t\eta^v_{ut} + (u_t)^2\eta^v_u + u_tv_t\eta^v_{uv}$$
$$+v_t\eta^u_{vt} + v_tu_t\eta^u_{vu} + (v_t)^2\eta^u_{vv} - v_x\xi^x_{tt} - v_xu_t\xi^x_{tu}$$
$$-v_xv_t\xi^x_{tv} - v_xu_t\xi^x_{ut} - v_x(u_t)^2\xi^x_u - v_xu_tv_t\xi^x_{uv}$$
$$-v_xv_t\xi^x_{vt} - v_xv_tu_t\xi^x_{vu} - v_x(v_t)^2\xi^x_{vv} - v_t\xi^t_{tt}$$
$$-u_tv_t\xi^t_{tu} - (v_t)^2\xi^t_{tv} - v_tu_t\xi^x_{ut} - (u_t)^2v_t\xi^x_u$$
$$-(v_t)^2u_t\xi^x_{uv} - (v_t)^2\xi^t_{vt} - u_t(v_t)^2\xi^t_{vu} - (v_t)^3\xi^t_{vv}$$
$$-v_{tx}(\xi^x_t + u_t\xi^x_u + v_t\xi^x_v) - v_{tt}(\xi^t_t + u_t\xi^t_u + v_t\xi^t_v)$$

If we substitute τ_7, τ_5. When we are solving the equations (determining equations), we write the equations as the polynomial of the derivatives of the dependent variables. Thus, we solve the equations by equating the coefficients to zero. We will satisfy the equality. So we have:

$$\eta^v_{ut} = 0, \eta^v_{ux} = 0, \eta^v_u = 0$$
$$\eta^v_{uv} + \eta^u_{uv} = 0$$
$$\eta^v_{vt} + \eta^u_{vt} = 0$$
$$\eta^v_{uv} + \eta^v_u = 0$$
$$-\xi^x_{tt} - \eta^v_{xv} - \eta^v_{ux} + \xi^x_{xx} = 0$$
$$\eta^v_{tt} - \eta^v_{xx} - \eta^u\cos(u-v) + \eta^v\cos u - v) = 0$$

We write the equation as the polynomial of the derivatives of u and v and set the coefficients equal to zero. One of the methods to be used for the solution of this system of equations is the series solution. By writing the infinitesimals as powers of independent variables, we substitute them in the determining equations we obtained before and look for solutions. While doing these we expand the sin and cos functions into power series.

If the infinitesimals are tried to be solved by extending the series in terms of the second order powers of the dependent and independent variables.

The determining equations are:

$$\xi^x = a_0 + a_{10}x + a_{11}t + a_{12}u + a_{13}v + a_{14}xt + a_{15}ux + a_{16}vx$$

$$+a_{17}x^2 + a_{18}ut + a_{19}vt + a_{20}t^2 + a_{21}uv + a_{22}u^2 + a_{23}v^2$$

$$\xi^t = b_0 + b_{10}x + b_{11}t + b_{12}u + b_{13}v + b_{14}xt + b_{15}ux + b_{16}vx$$
$$+b_{17}x^2 + b_{18}ut + b_{19}vt + b_{20}t^2 + b_{21}uv + b_{22}u^2 + b_{23}v^2$$

$$\eta^u = c_0 + c_{10}x + c_{11}t + c_{12}u + c_{13}v + c_{14}xt + c_{15}ux + c_{16}vx$$
$$+c_{17}x^2 + c_{18}ut + c_{19}vt + c_{20}t^2 + c_{21}uv + c_{22}u^2 + c_{23}v^2$$

$$\eta^v = d_0 + d_{10}x + d_{11}t + d_{12}u + d_{13}v + d_{14}xt + d_{15}ux + d_{16}vx$$
$$+d_{17}x^2 + d_{18}ut + d_{19}vt + d_{20}t^2 + d_{21}uv + d_{22}u^2 + d_{23}v^2$$

After substituting the polynomials we found in the equation, we were able to write the coefficients in terms of each other. This allowed us to reduce the number of coefficients.

$$a_{10} = a_{11}, a_{12} = 2a_{13} = 2a_{14}$$

$$c_0 = d_{0,}c_{10} = d_{10}, c_{11} = d_{11}, c_{12} = d_{12}, c_{13} = d_{13}, c_{14} = d_{14}$$

$$c_{15} = d_{15}, c_{16} = d_{16}, c_{17} = d_{17}, c_{20} = d_{20}$$

$$\xi^x = a_0 + a_{10}x + a_{10}t + 2a_{13}xt + a_{13}x^2 + a_{13}t^2$$

$$\xi^t = b_0$$

$$\eta^u = c_0 + c_{10}x + c_{11}t + c_{12}u + c_{13}v + c_{14}xt + c_{16}vx + c_{17}x^2 + c_{20}t^2$$

$$\eta^v = c_0 + c_{10}x + c_{11}t + c_{12}u + c_{13}v + c_{14}xt + c_{16}vx + c_{17}x^2 + c_{20}t^2$$

In the determining equations, we substitute them and set the coefficients to zero. Because the equations must be satisfied for each u and v. *A,B,C,D,E,F,G,H,I,J,K,L,M* are arbitrary functions.

$$X = (A + Bx + Bt + 2Cxt + Cx^2 + Ct^2)\frac{\partial}{\partial x}$$

$$+D\frac{\partial}{\partial t}$$

$$+(E+Fx+Gt+Hu+Iv+Jxt+Kvx+Lx^2+Mt^2)\frac{\partial}{\partial u}$$

$$+(E+Fx+Gt+Hu+Iv+Jxt+Kvx+Lx^2+Mt^2)\frac{\partial}{\partial v}$$

We set any 12 of the 13 consants to zero. Then, we have got a basis of symmetry groups [10]:

$$X_1=\frac{\partial}{\partial x},\ X_2=(x+t)\frac{\partial}{\partial t},\ X_3=(2xt+x^2+t^2)\frac{\partial}{\partial x},\ X_4=\frac{\partial}{\partial t} \tag{3.6}$$

$$X_5=\frac{\partial}{\partial u}+\frac{\partial}{\partial v},\ X_6=x\frac{\partial}{\partial u}+x\frac{\partial}{\partial v},\ X_7=t\frac{\partial}{\partial u}+t\frac{\partial}{\partial v} \tag{3.7}$$

$$X_8=u\frac{\partial}{\partial u}+u\frac{\partial}{\partial v},\ X_9=v\frac{\partial}{\partial u}+v\frac{\partial}{\partial v},\ X_{10}=xt\frac{\partial}{\partial u}+xt\frac{\partial}{\partial v} \tag{3.8}$$

$$X_{11}=vx\frac{\partial}{\partial u}+vx\frac{\partial}{\partial v},\ X_{12}=x^2\frac{\partial}{\partial u}+x^2\frac{\partial}{\partial v} \tag{3.9}$$

$$X_{13}=t^2\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial v} \tag{3.10}$$

Thus, we obtain thirteen families of transformations:

$$u^*=u, v^*=v, x^*=x+a, t^*=t$$

$$u^*=u, v^*=v, x^*=e^a(x+t)-t, t^*=t$$

$$u^*=u, v^*=v, x^*=\frac{x+at(x+t)}{1-a(x+t)}, t^*=t$$

$$u^*=u, v^*=v, x^*=x+a, t^*=t+a$$

$$u^*=u+a, v^*=v+a, x^*=x+a, t^*=t$$

$$u^*=u+ax, v^*=v+ax, x^*=x+a, t^*=t$$

$$u^*=u+at, v^*=v+at, x^*=x+a, t^*=t$$

$$u^*=e^au, v^*=v+au, x^*=x+a, t^*=t$$

$$u^*=u+av, v^*=e^av, x^*=x+a, t^*=t$$

$$u^*=u+xta, v^*=v+xta, x^*=x+a, t^*=t$$

$$u^* = u + vxa, v^* = e^{ax}v, x^* = x + a, t^* = t$$

$$u^* = u + ax^2, v^* = v + ax^2, x^* = x + a, t^* = t$$

$$u^* = u + at^2, v^* = v + at^2, x^* = x + a, t^* = t$$

4
RESULTS AND DISCUSSION

The Lie symmetry method is a very functional and useful method for systems of differential equations. This method covers many existing methods, in other words, collaborates them. We can think of symmetries as a tool, and this tool has many applications in solving systems of differential equations. Our aim may not always be to solve differential equations. We can classify equations according to their character or purpose using Lie symmetries. Such as being linear, homogeneous, integrable.

Equivalent solutions are obtained under invariant conditions. It means finding other equivalent solutions from known results. We can say that the situation is different for ODEs and PDEs. Invariant solutions for ODEs are found by solving algebraic equations, which are the reduced equation of the main equation. For PDEs, invariant solutions can be found after reduction of the independent variables. Although the Lie symmetry of systems of partial differential equations was not fully implemented in the first years, it is a work that gained momentum after a while. For systems of systems of differential equations, we apply the infinitesimal generator if we have how many equations, similarly as if we were solving differential equations separately.

In this thesis we have shown that it can be applied to some remarkable equations and systems of PDEs. The main model of our study is coupled system of sine-Gordon equations. They are nonlinear, inhomogeneous equations and have very important influences in physics and biology. We investigate symmetry transformations and obtain invariance criterions for the corresponding Lie symmetry transformations. Our process of finding equivalent solutions is based on an algorithm where we use infinitesimal generators. It was observed that the sine-Gordon equations also preserve these symmetries under some assumptions. We applied the definition and conditions of invariance for equations.

In Lie's studies, continuous groups are determined by infinitesimal operators(generators). Given the infinitesimal operators, we solve the Lie equations using the infinitesimals and obtain the group transformations. A base of Lie algebra

corresponding to the Lie symmetry group since it is a linear combination, this base n can be combined with any element. similarity solution can be obtained. Any element of this base n is Lie since the algebra is a one-dimensional subalgebra, this similarity solution is one-dimensional family of all similarity solutions formed by subalgebras with any transformation to another similarity solution in the set profit may come [14].

The present study has only explored the point transformations of the reviewed model, contains some part of the solutions of the sine-Gordon equations. Some results were revealed using the power series expansion. Since the equations are not homogeneous, we proceeded by using the power series expansion. In our further studies, we will work with inhomogeneous equations with another perspective. We will concentrate on the generalized symmetries of PDE systems.

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PUBLICATIONS FROM THE THESIS

Conference Papers

1. O. Yildirim and M. Mendi, "Lie Symmetry analysis of the coupled system of sine-Gordon equations", "International Conference of Young Mathematicians, 3-5 June 2021, Institute of Mathematics of NAS, Kyiv, Ukraine