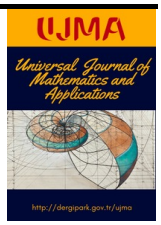
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Research Article

Trapezoid Inequalities for Twice Differentiable Mappings

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Abstract

This study introduces a new identity to derive alternative results for the inequality in (1.1) and compares them with previous findings. Additionally, new trapezoid inequalities are developed using the convexity of second-order derivatives' absolute values.

1. Introduction

The trapezoid inequality is a fundamental result in mathematical analysis and numerical integration, providing an estimate for the error in the trapezoidal rule when approximating definite integrals. It has been widely studied and generalized due to its close connections with properties such as convexity and differentiability.
Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function with bounded second derivative. The classical trapezoid inequality is given by

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{\|f''\|_\infty}{12}(b-a)^2.$$

More general forms involve bounds in terms of Lebesgue norms of the second derivative (see [1–3]):

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\begin{cases}\dfrac{(b-a)^2}{12}\left\|f''\right\|_\infty, & f''\in L_\infty[a,b],\\ \dfrac{[B(q+1,q+1)]^{1/q}(b-a)^{1+1/q}}{2}\left\|f''\right\|_p, & f''\in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1,\\ \dfrac{b-a}{8}\left\|f''\right\|_1, & f''\in L_1[a,b],\end{cases}\tag{1.1}$$

where $B(\cdot,\cdot)$ denotes the Beta function:

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}dt,\quad m,n>0.$$

These results are typically obtained using the identity ([4,5])

$$\frac{1}{b-a}\int_a^b(b-t)(t-a)f''(t)\,dt=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx,$$

together with Hölder's inequality.
In this study, we establish a new identity that leads to an alternative form of the trapezoid inequality, differing from the classical approach. Using this identity, we derive new bounds that can offer sharper or complementary estimates. Furthermore, we introduce novel inequalities based on the convexity of the absolute value of the second derivative, extending the scope of known results (similar to [6–10]). These findings contribute both to the theoretical development and to potential applications in numerical integration ([11–13]).

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2. Trapezoid Inequalities

In this section, we present our main results beginning with the following identity.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $\mathring{I}$ such that $f'' \in L_1[a,b]$, where $a,b \in I$ and $a < b$. Then the following identity holds:*

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b R_x(t)f''(t)\,dt\,dx = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx, \tag{2.1}$$

where

$$R_x(t) = \begin{cases} \frac{1}{2}(t-a)^2, & a \le t \le x, \\ \frac{1}{2}(t-b)^2, & x < t \le b. \end{cases}$$

Proof. Using the definition of $R_x(t)$ and integration by parts, we write:

$$\begin{aligned}\int_a^b R_x(t)f'(t)\,dt &= \frac{1}{2}\int_a^x (t-a)^2 f'(t)\,dt + \frac{1}{2}\int_x^b (t-b)^2 f'(t)\,dt \\ &= (b-a)\left(x - \frac{a+b}{2}\right)f(x) - \int_a^b P_x(t)f(t)\,dt,\end{aligned} \tag{2.2}$$

where

$$P_x(t) = \begin{cases} t-a, & a \le t \le x, \\ t-b, & x < t \le b. \end{cases}$$

Applying identity (2.2) to f', we obtain

$$(b-a)\left(x - \frac{a+b}{2}\right)f'(x) - \int_a^b P_x(t)f'(t)\,dt = \int_a^b R_x(t)f''(t)\,dt.$$

Integrating both sides with respect to x over $[a,b]$, we get

$$(b-a)\int_a^b \left(x - \frac{a+b}{2}\right)f'(x)\,dx - \int_a^b\int_a^b P_x(t)f'(t)\,dt\,dx = \int_a^b\int_a^b R_x(t)f''(t)\,dt\,dx.$$

Applying integration by parts to the first term and changing the order of integration in the second term, we arrive at

$$\frac{1}{b-a}\int_a^b\int_a^b R_x(t)f''(t)\,dt\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right] - \int_a^b f(x)\,dx.$$

Finally, noting that

$$\frac{1}{b-a}\int_a^b\int_a^b P_x(t)f'(t)\,dt\,dx = 0,$$

the identity (2.1) follows. □

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I such that $f \in L_1[a,b]$, where $a,b \in I$ and $a < b$. Then the following inequalities hold:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \begin{cases} \frac{(b-a)^2}{12}\|f''\|_\infty, & f'' \in L_\infty[a,b], \\ \frac{2^{\frac{1}{q}-1}(b-a)^{\frac{2}{q}}}{(2q+1)^{1/q}(2q+2)^{1/q}}\|f''\|_p, & f'' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ \frac{(b-a)}{8}\|f''\|_1, & f'' \in L_1[a,b]. \end{cases}$$

Here, $\|\cdot\|$ denotes the usual Lebesgue norms:

$$\|h\|_\infty := ess\sup_{t\in[a,b]}|h(t)|, \quad \|h\|_p := \left(\int_a^b |h(t)|^p\,dt\right)^{1/p}, \quad 1 \le p < \infty.$$

Proof. By Lemma 2.1 and the properties of the modulus, we get:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{(b-a)^2}\int_a^b\int_a^b |R_x(t)|\,|f''(t)|\,dt\,dx. \tag{2.3}$$

Case 1: $f'' \in L_\infty[a,b]$.
We estimate:

$$\int_a^b\int_a^b |R_x(t)|\,dt\,dx = \int_a^b\int_a^x \frac{(t-a)^2}{2}\,dt\,dx + \int_a^b\int_x^b \frac{(t-b)^2}{2}\,dt\,dx = \frac{(b-a)^4}{12}.$$

Thus,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{12}\|f''\|_\infty.$$

Case 2: $f'' \in L_p[a,b]$, $p>1$.
By Hölder's inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{\|f''\|_p}{(b-a)^2}\left(\int_a^b\int_a^b |R_x(t)|^q\,dt\,dx\right)^{1/q}.$$

We compute:

$$\int_a^b\int_a^x \frac{(t-a)^{2q}}{2^q}\,dt\,dx+\int_a^b\int_x^b \frac{(t-b)^{2q}}{2^q}\,dt\,dx=\frac{2(b-a)^{2q+2}}{2^q(2q+1)(2q+2)}.$$

This gives the second bound.
Case 3: $f'' \in L_1[a,b]$.
Using the property $\int |f(t)|g(t)dt \le \sup|f(t)|\int|g(t)|dt$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{(b-a)^2}\left(\sup_{t\in[a,b]}\int_a^b |R_x(t)|dx\right)\|f''\|_1.$$

The total magnitude of the integrated kernel is computed as:

$$\sup_{t\in[a,b]}\left(\int_a^b |R_x(t)|dx\right)=\frac{(b-a)^3}{8}.$$

Substituting the supremum and simplifying the factor $(b-a)$:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{(b-a)^2}\cdot\frac{(b-a)^3}{8}\|f''\|_1=\frac{b-a}{8}\|f''\|_1.$$

□

Theorem 2.3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I such that $f\in L_1[a,b]$ for $a,b\in I$, $a<b$. If $|f''|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{12}\left(\frac{|f''(a)|+|f''(b)|}{2}\right).$$

Proof. From Lemma 2.1 and using the definition of $R_x(t)$ along with standard properties of integrals and the modulus, we obtain:

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{1}{2(b-a)^2}\int_a^b\int_a^x (t-a)^2\left|f''(t)\right|dt\,dx\\ &\quad+\frac{1}{2(b-a)^2}\int_a^b\int_x^b (b-t)^2\left|f''(t)\right|dt\,dx\\ &=\frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\left|f''(t)\right|dt.\end{aligned}$$

Using the convexity of $|f''(t)|$, we write

$$\left|f''(t)\right| \le \frac{b-t}{b-a}\left|f''(a)\right|+\frac{t-a}{b-a}\left|f''(b)\right|.$$

Thus,

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le \frac{1}{2(b-a)^2}\int_a^b (b-t)(t-a)\left[\frac{b-t}{b-a}\left|f''(a)\right|+\frac{t-a}{b-a}\left|f''(b)\right|\right]dt\\ &=\frac{1}{2(b-a)^3}\int_a^b\left[(t-a)(b-t)^2\left|f''(a)\right|+(t-a)^2(b-t)\left|f''(b)\right|\right]dt\\ &=\frac{b-a}{12}\left(\frac{|f''(a)|+|f''(b)|}{2}\right),\end{aligned}$$

which completes the proof. □

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I such that $f \in L_1[a,b]$ for $a,b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$ for some $q > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^{1+\frac{1}{p}}}{2}\left(\frac{\Gamma(p+1)^2}{\Gamma(2p+2)}\right)^{1/p} \cdot \left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From the estimate in (2.3) and using Hölder's inequality, we get:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2(b-a)}\left(\int_a^b (b-t)^p (t-a)^p\,dt\right)^{1/p}\left(\int_a^b |f''(t)|^q\,dt\right)^{1/q}.$$

By convexity of $|f''|^q$ and the substitution $t = (1-u)a + ub$, we obtain:

$$\begin{aligned}\int_a^b |f''(t)|^q\,dt &\leq (b-a)\int_0^1 \left[(1-u)|f''(a)|^q + u|f''(b)|^q\right]du \\ &= (b-a)\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right).\end{aligned}$$

Also,

$$\int_a^b (b-t)^p (t-a)^p\,dt = (b-a)^{2p+1}\int_0^1 u^p (1-u)^p\,du = (b-a)^{2p+1}\cdot\frac{\Gamma(p+1)^2}{\Gamma(2p+2)}.$$

Thus,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{1}{2(b-a)}(b-a)^{2+\frac{1}{p}}\left(\frac{\Gamma(p+1)^2}{\Gamma(2p+2)}\right)^{1/p}\cdot\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{1/q} \\ &= \frac{(b-a)^{1+\frac{1}{p}}}{2}\left(\frac{\Gamma(p+1)^2}{\Gamma(2p+2)}\right)^{1/p}\cdot\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{1/q},\end{aligned}$$

which completes the proof. □

3. Conclusion

We have established new integral identities and inequalities for twice differentiable functions whose second derivatives belong to various Lebesgue spaces. Starting from a kernel-based integral identity, we derived bounds on the difference between the average of endpoint values and the integral mean in terms of the second derivative norms. Moreover, assuming convexity of the absolute value or power of the second derivative, sharper estimates depending only on endpoint values were obtained. These results provide useful tools for bounding integral deviations and may be extended to more general function classes in future work.

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