Journal of Applied Mathematics and Computing (2023) 69:4045–4068
https://doi.org/10.1007/s12190-023-01915-2

ORIGINAL RESEARCH

Additive double polycyclic codes over $\mathbb{F}_{p^2}$ and their applications to quantum codes

Mustafa Sarı[1]

Received: 23 May 2023 / Revised: 3 August 2023 / Accepted: 30 August 2023 /
Published online: 12 September 2023
© The Author(s) under exclusive licence to Korean Society for Informatics and Computational Applied Mathematics 2023

Abstract
In this paper, we introduce additive double polycyclic codes over $\mathbb{F}_{p^2}$ for any prime p. We study their algebraic structures and determine generator polynomials of these codes. We also give cardinality of them. Additionally, we establish that both Euclidean dual and Hermitian dual of an additive double polycyclic code over $\mathbb{F}_{p^2}$ are also an additive double polycyclic code over $\mathbb{F}_{p^2}$. As practical applications of these codes, we exhibit additive double polycyclic codes over $\mathbb{F}_9$ which contain triple as many codewords as optimal linear codes of same length and minimum distance over $\mathbb{F}_9$, and we derive optimal ternary linear codes and quantum codes from additive double polycyclic codes over $\mathbb{F}_9$.

Keywords Additive codes · Polycyclic codes · Quantum codes

Mathematics Subject Classification 94B05 · 94B60 · 81P70

1 Introduction

Additive codes over finite fields and other algebras have intensively been studied by researchers since they enable to do search on parameters of the codes due to their rich algebraic structures [1–20]. Calderbank et al. studied on additive codes over $\mathbb{F}_4$ and explored that there exists relation between additive codes over $\mathbb{F}_4$ and binary quantum codes [1]. As a generalization of [1], Ketkar et al. considered additive codes over $\mathbb{F}_q$ to construct nonbinary quantum codes [2]. Aydoğdu and Siap studied the algebraic structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes as a generalization binary and quaternary codes [3]. In [5], the authors extended the results in [3] on $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ for any prime p. In [7], Borges et al. determined the generator polynomials of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes and their

✉ Mustafa Sarı
musari@yildiz.edu.tr

[1] Department of Mathematics, Yildiz Technical University, 34220 Esenler, Istanbul, Turkey

duals, and in [8], they carried the findings on $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes over $\mathbb{Z}_{p^s}\mathbb{Z}_{p^s}$-additive cyclic codes for any prime p. Shi et al. showed how to construct asymptotically good additive constacyclic codes over extension of fields with a fixed degree [9]. In [10], Shi et al. proved the weight enumerators of the images of a $\mathbb{Z}_p\mathbb{Z}_{p^k}$-additive code and its dual are formally dual. Aydoğdu and Gürsoy determined generator polynomials and minimal spanning sets for $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive cyclic codes [11]. In [12], the author extended the results in [11] on $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}Z_{p^t}$. Shi et al. put forward some binary optimal linear codes obtained from $\mathbb{Z}_2\mathbb{Z}_4$-additive codes in accordance with Griesmer bound [13].

More recently, Shi et al. considered double polycirculant codes as a generalization of double circulant codes [14]. Wu and Shi considered $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes and studied their duals under a proper inner product [15]. Wu et al. generalized 1-quasi cyclic codes over finite fields to finite chain rings [16]. Shi et al. studied additive cyclic and complementary dual codes of odd length over $\mathbb{F}_4$ with respect to both the trace Hermitian inner product and the Euclidean inner product [17]. Karbaski et al. examined the structure and properties of additive right and left polycyclic codes over $\mathbb{F}_4$ induced by a binary vector in $\mathbb{F}_2^n$ and presented some additive right polycyclic codes having more codewords than optimal binary linear codes [18]. Sarı and Köroğlu generalized the findings obtained in [18] on $\mathbb{F}_{p^2}$ for any prime p [19]. By inspired above studies, we consider additive double polycyclic codes over $\mathbb{F}_{p^2}$ for the vectors over $\mathbb{F}_p$ and apply these codes to quantum codes.

This paper consists of six sections designed as follows: In Sect. 2, the basic definitions and notations are given. In Sect. 3, for two vectors a and b in $\mathbb{F}_p^n$, additive (a, b)-double polycyclic codes over $\mathbb{F}_{p^2}$ are introduced and generator polynomials of these codes are given. Several additive double polycyclic codes, which contain three times as many codewords as optimal linear codes of same length and minimum distance over $\mathbb{F}_9$, are listed in a table. Furthermore, numerous optimal ternary linear codes obtained from additive double polycyclic codes over $\mathbb{F}_9$ are presented. In Sect. 4, Euclidean and Hermitian duals of additive (a, b)-double polycyclic codes over $\mathbb{F}_{p^2}$ are studied. In Sect. 5, a table presenting the parameters of several ternary quantum codes derived from additive double polycyclic codes over $\mathbb{F}_9$ is given. In Sect. 6, the paper is concluded.

2 Preliminaries

Let $\mathbb{F}_p$ be a base field and $\mathbb{F}_q$ be a field extension of $\mathbb{F}_p$. Denote $\mathbb{F}_q^n$ to be the set of all n-tuples over $\mathbb{F}_q$. A code over the alphabet $\mathbb{F}_q$ of length n is a nonempty set of $\mathbb{F}_q^n$. An additive code over $\mathbb{F}_q$ of length n is a nonempty set of $\mathbb{F}_q^n$ which is closed under componentwisely addition of vectors in that set. A linear code over $\mathbb{F}_q$ of length n is a subspace of the vector space $\mathbb{F}_q^n$. The (Hamming) distance between two vectors is the number of their distinct components. The minimum (Hamming) distance for a code C is the minimum (Hamming) distance between the distinct vectors in this code. An $(n, K, d)_q$ code means a code C over $\mathbb{F}_q$ of length n, minimum distance d and size $|C| = K$. If C is a linear code over $\mathbb{F}_q$ of length n, minimum distance d and dimension k, then it is denoted by $[n, k, d]_q$. For a code C over $\mathbb{F}_q$ of length n, there

exists its (Euclidean) dual code $C^{\perp} = \left\{ x \in \mathbb{F}_q^n : \langle x, y \rangle = \sum_{i=0}^{n-1} x_i y_i = 0, \ \forall y \in C \right\}$. Note that $C^{\perp}$ is a linear code over $\mathbb{F}_q$ of dimension $n-k$ if C is a linear code over $\mathbb{F}_q$ of length n and dimension k. Let $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_q^n$ and C be a linear code over $\mathbb{F}_q$ of length n. Then, C is a polycyclic code induced by the vector a if $(c_{n-1}a_0, c_0 + c_{n-1}a_1, \ldots, c_{n-2} + c_{n-1}a_{n-1}) \in C$ for all $(c_0, c_1, \ldots, c_{n-1}) \in C$. In particular, C is called λ-constacyclic if $a = (\lambda, 0, \ldots, 0) \in \mathbb{F}_q^n$ and C is called cyclic code if $a = (1, 0, \ldots, 0) \in \mathbb{F}_q^n$. Let $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$ be polynomial representation of the vector $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_q^n$. Then, there exists a one-to-one correspondence between polycyclic codes induced by the vector a over $\mathbb{F}_q$ of length n and $\mathbb{F}_p[x]$-submodules of $\frac{\mathbb{F}_q[x]}{\langle x^n - a(x) \rangle}$. Since every $\mathbb{F}_p[x]$-submodule of $\frac{\mathbb{F}_q[x]}{\langle x^n - a(x) \rangle}$ is uniquely generated, for each polycyclic code C by the vector a over $\mathbb{F}_q$ of length n, there exists a generator polynomial $g(x)$ dividing $x^n - a(x)$ such that $C = \langle g(x) \rangle$. If $\deg(g(x)) = k$, then C is a linear code of $n-k$ dimensional.

3 Additive double polycyclic codes over $\mathbb{F}_{p^2}$

Let $\mathbb{F}_{p^2}^n$ be the set of n-tuples over $\mathbb{F}_{p^2}$. An additive code over $\mathbb{F}_{p^2}$ of length n is a subset of $\mathbb{F}_{p^2}^n$ which is closed under addition, in other words, an additive subgroup of $\mathbb{F}_{p^2}^n$. Throughout the rest of this paper, ω denotes a primitive $\left(p^2 - 1\right)$-root of unity in $\mathbb{F}_{p^2}$ unless otherwise stated.

Definition 1 Let $a = (a_0, a_1, \ldots, a_{l-1})$ and $b = (b_0, b_1, \ldots, b_{m-1})$ be two vectors in $\mathbb{F}_p^n$. An additive code C over $\mathbb{F}_{p^2}$ of length $n = l+m$ is called an additive (a, b)-double polycyclic code if

$$\begin{aligned}&(d_{l-1}a_0, d_0 + d_{l-1}a_1, \ldots, d_{l-2} + d_{l-1}a_{l-1} |\\ &\quad t_{m-1}b_0, t_0 + t_{m-1}b_1, \ldots, t_{m-2} + t_{m-1}b_{m-1}) \in C\end{aligned}$$

whenever $(d_0, d_1, \ldots, d_{l-1} | t_0, t_1, \ldots, t_{m-1}) \in C$.

Let $a(x)$ and $b(x)$ be two polynomials over $\mathbb{F}_p$. Denote the set $R_{l,m}$ as the cartesian product of quotient rings $R_{l,m} = \frac{\mathbb{F}_{p^2}[x]}{\langle x^n - a(x) \rangle} \times \frac{\mathbb{F}_{p^2}[x]}{\langle x^m - b(x) \rangle}$. Then, $R_{l,m}$ is an $\mathbb{F}_p[x]$-module with the addition

$$(d_1(x), t_1(x)) + (d_2(x), t_2(x)) = (d_1(x) + d_2(x), t_1(x) + t_2(x))$$

and the multiplication

$$a(x)(d(x), t(x)) = (a(x)d(x), a(x)t(x))$$

where $a(x) \in \mathbb{F}_p[x]$.

The map $\phi : \mathbb{F}_{p^2}^l \times \mathbb{F}_{p^2}^m \to R_{l,m}$ given by

$$\begin{aligned}\mathbf{c} &= (d_0, d_1, \ldots, d_{l-1} | \, t_0, t_1, \ldots, t_{m-1}) \mapsto \\ \mathbf{c}(x) &= \left(d_0 + d_1 x + \cdots + d_{l-1} x^{l-1}, t_0 + t_1 x + \cdots + t_{m-1} x^{m-1}\right) \\ &= (d(x), t(x))\end{aligned}$$

is bijective. Then, we have the following theorem.

Theorem 1 *There exists one-to-one correspondence between additive (a, b)-double polycyclic codes over $\mathbb{F}_{p^2}$ of length $n = l + m$ and $\mathbb{F}_p[x]$-submodules of $R_{l,m}$.*

Proof Let C be an additive (a, b)-double polycyclic code over $\mathbb{F}_{p^2}$ of length $n = l+m$. Clearly, $\phi(C)$ is closed under addition. To show that $\phi(C)$ is closed under multiplication by all polynomials in $\mathbb{F}_p[x]$, it is enough to prove that $x(d(x), t(x)) \in \phi(C)$ for all $(d(x), t(x)) \in \phi(C)$. Let the polynomial representation of $\mathbf{c} = (d_0, d_1, \ldots, d_{n-1} | \, t_0, t_1, \ldots, t_{n-1}) \in C$ be $\mathbf{c}(x) = (d(x), t(x))$. Since $x^l = a(x)$ and $x^m = b(x)$, we get

$$\begin{aligned}x(d(x), t(x)) &= (xd(x), xt(x)) \\ &= \Big(d_{l-1}a_0 + (d_{l-1}a_1 + d_0)x + \cdots + (d_{l-1}a_{l-1} + d_{l-2})x^{l-1}, \\ &\quad t_{m-1}b_0 + (t_{m-1}b_1 + t_0)x + \cdots + (t_{m-1}b_{m-1} + t_{m-2})x^{m-1}\Big).\end{aligned}$$

Since C is an additive (a, b)-double polycyclic code,

$$\begin{aligned}&(d_{l-1}a_0, d_0 + d_{l-1}a_1, \ldots, d_{l-2} + d_{l-1}a_{l-1} | \\ &\quad t_{m-1}b_0, t_0 + t_{m-1}b_1, \ldots, t_{m-2} + t_{m-1}b_{m-1}) \in C\end{aligned}$$

and so its polynomial representation

$$\begin{aligned}&\Big(d_{l-1}a_0 + (d_{l-1}a_1 + d_0)x + \cdots + (d_{l-1}a_{l-1} + d_{l-2})x^{l-1}, \\ &\quad t_{m-1}b_0 + (t_{m-1}b_1 + t_0)x + \cdots + (t_{m-1}b_{m-1} + t_{m-2})x^{m-1}\Big) \in \phi(C).\end{aligned}$$

The converse is clear. □

Theorem 1 enables to view an additive (a, b)-double polycyclic code over $\mathbb{F}_{p^2}$ as an $\mathbb{F}_p[x]$-submodule of $R_{l,m}$, and so we need to study the algebraic structure of $\mathbb{F}_p[x]$-submodules of $R_{l,m}$ to determine the generators of additive (a, b)-double polycyclic codes over $\mathbb{F}_{p^2}$.

Let C be an additive (a, b)-double polycyclic code over $\mathbb{F}_{p^2}$ of length $n = l + m$. Define the map

$$\begin{aligned}\varphi : C \leq \frac{\mathbb{F}_{p^2}[x]}{\langle x^l - a(x)\rangle} \times \frac{\mathbb{F}_{p^2}[x]}{\langle x^m - b(x)\rangle} &\to \frac{\mathbb{F}_{p^2}[x]}{\langle x^l - a(x)\rangle}, \\ (f_1(x), f_2(x)) &\mapsto f_1(x).\end{aligned}$$

Since φ is an $\mathbb{F}_p[x]$-module homomorphism, $\varphi(C)$ is an $\mathbb{F}_p[x]$-submodule of $\frac{\mathbb{F}_{p^2}[x]}{\langle x^l - a(x)\rangle}$ and $Ker(\varphi)$ is an $\mathbb{F}_p[x]$-submodule of the code C. Then, from [19], we can write

$$\varphi(C) = \langle \omega g_1(x) + u_1(x), t_1(x)\rangle$$

where $g_1(x), u_1(x)$ and $t_1(x)$ are polynomials in $\mathbb{F}_p[x]$ satisfying the conditions 1 and 2 given in [19] with $g_1(x), t_1(x) \mid \left(x^l - a(x)\right)$, $\deg(u_1(x)) < \deg(t_1(x))$ and $t_1(x) \mid \left(\frac{x^l - a(x)}{g_1(x)} u_1(x)\right)$. See that

$$Ker(\varphi) = \left\{(0, f_2(x)) \in C : f_2(x) \in \frac{\mathbb{F}_{p^2}[x]}{\langle x^m - b(x)\rangle}\right\}.$$

Then, the set

$$J = \left\{f_2(x) \in \frac{\mathbb{F}_{p^2}[x]}{\langle x^m - b(x)\rangle} : (0, f_2(x)) \in Ker(\varphi)\right\}$$

is an $\mathbb{F}_p[x]$-submodule of $\frac{\mathbb{F}_{p^2}[x]}{\langle x^m - b(x)\rangle}$, and so from [19], we have

$$J = \langle \omega g_2(x) + u_2(x), t_2(x)\rangle,$$

where $g_2(x), u_2(x)$ and $t_2(x)$ are polynomials in $\mathbb{F}_p[x]$ satisfying the conditions 1 and 2 given in [19] with $g_2(x), t_2(x) \mid (x^m - b(x))$, $\deg(u_2(x)) < \deg(t_2(x))$ and $t_2(x) \mid \left(\frac{x^m - b(x)}{g_2(x)} u_2(x)\right)$. For each element $(0, f_2(x)) \in Ker(\varphi)$,

$$f_2(x) \in J = \langle \omega g_2(x) + u_2(x), t_2(x)\rangle$$

and so, for some $q_1(x), q_2(x) \in \mathbb{F}_p[x]$,

$$f_2(x) = q_1(x)(\omega g_2(x) + u_2(x)) + q_2(x) t_2(x),$$

which implies that

$$(0, f_2(x)) = q_1(x)(0, \omega g_2(x) + u_2(x)) + q_2(x)(0, t_2(x)).$$

Therefore, $Ker(\varphi)$ is an $\mathbb{F}_p[x]$-submodule of C of the form

$$Ker(\varphi) = \langle (0, \omega g_2(x) + u_2(x)), (0, t_2(x))\rangle.$$

It follows from the first isomorphism theorem for modules that

$$\frac{C}{Ker(\varphi)} \cong \varphi(C) = \langle \omega g_1(x) + u_1(x), t_1(x)\rangle.$$

Let $(\omega g_1(x)+u_1(x), p_1(x)), (t_1(x), p_2(x)) \in C$ such that

$$\varphi(\omega g_1(x)+u_1(x), p_1(x)) = \omega g_1(x)+u_1(x)$$
$$\varphi(t_1(x), p_2(x)) = t_1(x).$$

As a consequence of above discuss, any additive (a,b)-double polycyclic code C over $\mathbb{F}_{p^2}$ of length $l+m$ can be generated as an $\mathbb{F}_p[x]$-submodule of $R_{l,m}$ having the form

$$C = \langle (\omega g_1(x)+u_1(x), p_1(x)), (t_1(x), p_2(x)), (0, \omega g_2(x)+u_2(x)), (0, t_2(x)) \rangle.$$

Since $p_1(x)$ and $p_2(x)$ are polynomials in $\mathbb{F}_{p^2}[x]$, we can rearrange C as the following:

$$C = \langle (\omega g_1(x)+u_1(x), \omega l_1(x)+l_2(x)), (t_1(x), \omega l_3(x)+l_4(x)), (0, \omega g_2(x)+u_2(x)), (0, t_2(x)) \rangle,$$

where $l_i(x)$ is a polynomial in $\mathbb{F}_p[x]$.

Lemma 1 *Let C be an additive (a,b)-double polycyclic code C over $\mathbb{F}_{p^2}$ of length $l+m$ where*

$$C = \langle (\omega g_1(x)+u_1(x), \omega l_1(x)+l_2(x)), (t_1(x), \omega l_3(x)+l_4(x)), (0, \omega g_2(x)+u_2(x)), (0, t_2(x)) \rangle.$$

Then, we may assume the followings:

1. $\deg l_2(x)$ *and* $\deg l_4(x)$ *are less than* $\deg t_2(x)$,
2. $\deg l_1(x)$ *and* $\deg l_3(x)$ *are less than* $\deg g_2(x)$,

Proof 1. If $\deg l_2(x) \geq \deg t_2(x)$, then $l_2(x) = v_1(x) t_2(x) + r_1(x)$ for some $v_1(x), r_1(x) \in \mathbb{F}_p[x]$ where $r_1(x) = 0$ or $\deg r_1(x) < \deg t_2(x)$. Let

$$D = \langle (\omega g_1(x)+u_1(x), \omega l_1(x)+r_1(x)), (t_1(x), \omega l_3(x)+l_4(x)), (0, \omega g_2(x)+u_2(x)), (0, t_2(x)) \rangle.$$

It follows from

$$(\omega g_1(x)+u_1(x), \omega l_1(x)+l_2(x)) = (\omega g_1(x)+u_1(x), \omega l_1(x)+r_1(x)) + v_1(x)(0, t_2(x)) \in D$$

and

$$(\omega g_1(x)+u_1(x), \omega l_1(x)+r_1(x)) = (\omega g_1(x)+u_1(x), \omega l_1(x)+l_2(x)) - v_1(x)(0, t_2(x)) \in C$$

that $C = D$, which implies that we may assume that $\deg l_2(x) < \deg t_2(x)$. The case $\deg l_4(x) < \deg t_2(x)$ is similar.

2. If $\deg l_1(x) \geq \deg g_2(x)$, then $l_1(x) = v_2(x) g_2(x) + r_2(x)$ for some $v_2(x), r_2(x) \in \mathbb{F}_p[x]$ where $r_2(x) = 0$ or $\deg r_2(x) < \deg g_2(x)$. Let

$$D = \langle (\omega g_1(x) + u_1(x), \omega r_2(x) + l_2(x) - v_2(x) u_2(x)), \\ (t_1(x), \omega l_3(x) + l_4(x))(0, \omega g_2(x) + u_2(x)), (0, t_2(x)) \rangle.$$

It follows from that

$$(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) = (\omega g_1(x) + u_1(x), \omega r_2(x) + l_2(x) - v_2(x) u_2(x)) \\ + v_2(x)(0, \omega g_2(x) + u_2(x)) \in D$$

and

$$(\omega g_1(x) + u_1(x), \omega r_2(x) + l_2(x) - v_2(x) u_2(x)) \\ = (\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) - v_2(x)(0, \omega g_2(x) + u_2(x)) \in C$$

that $C = D$, which implies that we may assume that $\deg l_1(x) < \deg g_2(x)$. The case $\deg l_3(x) < \deg g_2(x)$ is similar.

□

Lemma 2 *Let C be an additive (a, b)-double polycyclic code over $\mathbb{F}_{p^2}$ of length $l + m$ where*

$$C = \langle (\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)), (t_1(x), \omega l_3(x) + l_4(x)), \\ (0, \omega g_2(x) + u_2(x)), (0, t_2(x)) \rangle.$$

Then, we have the followings:

1. $g_1(x), t_1(x) \mid \left(x^l - a(x)\right)$, $\deg(u_1(x)) < \deg(t_1(x))$ *and* $t_1(x) \mid \left(\frac{x^l - a(x)}{g_1(x)} u_1(x)\right)$,
2. $g_2(x), t_2(x) \mid (x^m - b(x))$, $\deg(u_2(x)) < \deg(t_2(x))$ *and* $t_2(x) \mid \left(\frac{x^m - b(x)}{g_2(x)} u_2(x)\right)$,
3. $g_2(x) \mid \left(\frac{x^l - a(x)}{t_1(x)}\right) l_3(x)$, *and if* $l_3(x) \mid (x^m - b(x))$, *then*

$$t_2(x) \mid \left(\frac{x^m - b(x)}{l_3(x)}\right)\left(\frac{x^l - a(x)}{t_1(x)}\right) l_4(x),$$

4. $g_2(x) \mid \left(\frac{x^l - a(x)}{g_1(x)}\right)\left(l_1(x) - \frac{u_1(x)}{t_1(x)} l_3(x)\right)$, *and if* $l_1(x) \mid (x^m - b(x))$, *then*

$$g_2(x) \mid \left(\frac{x^m - b(x)}{l_1(x)}\right)\left(\frac{x^l - a(x)}{g_1(x)}\right)\frac{u_1(x) l_3(x)}{t_1(x)}.$$

Proof We already have *1* and *2* from [19]. Since

$$\left(\frac{x^l - a(x)}{t_1(x)}\right)(t_1(x), \omega l_3(x) + l_4(x))$$
$$= \left(0, \frac{x^l - a(x)}{t_1(x)}(\omega l_3(x) + l_4(x))\right) \in Ker(\varphi),$$
$$\left(\frac{x^l - a(x)}{t_1(x)}\right)(\omega l_3(x) + l_4(x)) \in J = \langle \omega g_2(x) + u_2(x), t_2(x)\rangle.$$

Since $u_2(x), t_2(x) \in \mathbb{F}_p[x]$ and J is an $\mathbb{F}_p[x]$-submodule, we get

$$g_2(x) \mid \left(\frac{x^l - a(x)}{t_1(x)}\right) l_3(x).$$

Moreover, if $l_3(x) \mid (x^m - b(x))$, then $\frac{x^m - b(x)}{l_3(x)} \in \mathbb{F}_p[x]$ and so

$$\left(\frac{x^m - b(x)}{l_3(x)}\right)\left(\frac{x^l - a(x)}{t_1(x)}\right)(\omega l_3(x) + l_4(x))$$
$$= \left(\frac{x^m - b(x)}{l_3(x)}\right)\left(\frac{x^l - a(x)}{t_1(x)}\right) l_4(x) \in J.$$

It follows from $\left(\frac{x^m - b(x)}{l_3(x)}\right)\left(\frac{x^l - a(x)}{t_1(x)}\right) l_4(x) \in \mathbb{F}_p[x]$ that

$$t_2(x) \mid \left(\frac{x^m - b(x)}{l_3(x)}\right)\left(\frac{x^l - a(x)}{t_1(x)}\right) l_4(x),$$

which completes the proof of *3*. See that

$$\left(\frac{x^l - a(x)}{g_1(x)}\right)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x))$$
$$= \left(\frac{x^l - a(x)}{g_1(x)} u_1(x), \frac{x^l - a(x)}{g_1(x)}(\omega l_1(x) + l_2(x))\right) \in C.$$

Since $t_1(x) \mid \left(\frac{x^l - a(x)}{g_1(x)} u_1(x)\right)$, say $k_1(x)\, t_1(x) = \left(\frac{x^l - a(x)}{g_1(x)}\right) u_1(x)$ for some $k_1(x) \in \mathbb{F}_p[x]$. Then, we get

$$\left(\frac{x^l - a(x)}{g_1(x)} u_1(x), \frac{x^l - a(x)}{g_1(x)}(\omega l_1(x) + l_2(x))\right) - k_1(x)(t_1(x), \omega l_3(x) + l_4(x))$$
$$= \left(0, \frac{x^l - a(x)}{g_1(x)}(\omega l_1(x) + l_2(x)) - k_1(x)(\omega l_3(x) + l_4(x))\right) \in Ker(\varphi)$$

and

$$\left(\frac{x^l - a(x)}{g_1(x)}\right)(\omega l_1(x) + l_2(x)) - k_1(x)(\omega l_3(x) + l_4(x)) \in J,$$

which implies that $g_2(x) \mid \left(\frac{x^l - a(x)}{g_1(x)}\right)\left(l_1(x) - \frac{u_1(x)}{t_1(x)} l_3(x)\right)$. If $l_1(x) \mid (x^m - b(x))$, then $\frac{x^m - b(x)}{l_1(x)} \in \mathbb{F}_p[x]$ and so

$$\left(\frac{x^m - b(x)}{l_1(x)}\right)\left(\left(\frac{x^l - a(x)}{g_1(x)}\right)(\omega l_1(x) + l_2(x)) - k_1(x)(\omega l_3(x) + l_4(x))\right)$$
$$= \left(\frac{x^m - b(x)}{l_1(x)}\right)\left(\frac{x^l - a(x)}{g_1(x)}\right) l_2(x) - \left(\frac{x^m - b(x)}{l_1(x)}\right) k_1(x)(\omega l_3(x) + l_4(x)) \in J,$$

which implies that $g_2(x) \mid \left(\frac{x^m - b(x)}{l_1(x)}\right)\left(\frac{x^l - a(x)}{g_1(x)}\right)\frac{u_1(x) l_3(x)}{t_1(x)}$. □

Lemma 3 *The generators of an additive (a, b)-double polycyclic code C over $\mathbb{F}_{p^2}$ of length $l + m$ where*

$$C = \langle (\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)), (t_1(x), \omega l_3(x) + l_4(x)), (0, \omega g_2(x) + u_2(x)), (0, t_2(x)) \rangle$$

are unique.

Proof Since $\varphi(C) = \langle \omega g_1(x) + u_1(x), t_1(x) \rangle$ and $J = \langle \omega g_2(x) + u_2(x), t_2(x) \rangle$ are $\mathbb{F}_p[x]$-submodules of $\frac{\mathbb{F}_{p^2}[x]}{\langle x^l - a(x) \rangle}$ and $\frac{\mathbb{F}_{p^2}[x]}{\langle x^m - b(x) \rangle}$ respectively, from Lemma III.IV in [19], the polynomials $g_i(x)$, $u_i(x)$ and $t_i(x)$ are unique. It remains to show that $\omega l_1(x) + l_2(x)$ and $\omega l_3(x) + l_4(x)$ are unique. Suppose that

$$\begin{aligned} C &= \langle (\omega g_1(x) + u_1(x), \omega l_{1,1}(x) + l_{2,1}(x)), (t_1(x), \omega l_{3,1}(x) + l_{4,1}(x)), \\ &\quad (0, \omega g_2(x) + u_2(x)), (0, t_2(x)) \rangle \\ &= \langle (\omega g_1(x) + u_1(x), \omega l_{1,2}(x) + l_{2,2}(x)), (t_1(x), \omega l_{3,2}(x) + l_{4,2}(x)), \\ &\quad (0, \omega g_2(x) + u_2(x)), (0, t_2(x)) \rangle. \end{aligned}$$

Then,

$$\begin{aligned} &(\omega g_1(x) + u_1(x), \omega l_{1,1}(x) + l_{2,1}(x)) - (\omega g_1(x) + u_1(x), \omega l_{1,2}(x) + l_{2,2}(x)) \\ &= (0, \omega(l_{1,1}(x) - l_{1,2}(x)) + l_{2,1}(x) - l_{2,2}(x)) \in Ker(\varphi) \end{aligned}$$

and so $\omega(l_{1,1}(x) - l_{1,2}(x)) + l_{2,1}(x) - l_{2,2}(x) \in J$ with $\deg(l_{1,1}(x) - l_{1,2}(x)) < \deg(g_2(x))$ and $\deg(l_{2,1}(x) - l_{2,2}(x)) < \deg(t_2(x))$. However, this contradicts with the condition 2 given in [19] unless $l_{1,1}(x) = l_{1,2}(x)$ and $l_{2,1}(x) = l_{2,2}(x)$. Therefore, $\omega l_{1,1}(x) + l_{2,1}(x) = \omega l_{1,2}(x) + l_{2,2}(x)$. The proof of the equality $\omega l_{3,1}(x) + l_{4,1}(x) = \omega l_{3,2}(x) + l_{4,2}(x)$ is similar. □

Let C be an additive (a, b)-double polycyclic code over $\mathbb{F}_{p^2}$ of length $l+m$ where

$$C = \langle (\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)), (t_1(x), \omega l_3(x) + l_4(x)), (0, \omega g_2(x) + u_2(x)), (0, t_2(x)) \rangle .$$

Since J is an $\mathbb{F}_p[x]$-submodule of $\frac{\mathbb{F}_{p^2}[x]}{\langle x^l - a(x) \rangle}$ and $\varphi(C)$ is an $\mathbb{F}_p[x]$-submodule of $\frac{\mathbb{F}_{p^2}[x]}{\langle x^m - b(x) \rangle}$, we deduce from $\frac{C}{Ker(\varphi)} \cong \varphi(C)$ and Theorem III.V in [19] that

$$|C| = |Ker(\varphi)| \, |\varphi(C)| = p^{l - \deg(g_1(x))} \cdot p^{l - \deg(t_1(x))} \cdot p^{m - \deg(g_2(x))} \cdot p^{m - \deg(t_2(x))}.$$

Now, define the following subsets of C:

$$S_1 = \bigcup_{i=0}^{l - \deg(g_1(x)) - 1} \left\{ x^i (\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) \right\},$$

$$S_2 = \bigcup_{i=0}^{l - \deg(t_1(x)) - 1} \left\{ x^i (t_1(x), \omega l_3(x) + l_4(x)) \right\},$$

$$S_3 = \bigcup_{i=0}^{m - \deg(g_2(x)) - 1} \left\{ x^i (0, \omega g_2(x) + u_2(x)) \right\},$$

$$S_4 = \bigcup_{i=0}^{m - \deg(t_2(x)) - 1} \left\{ x^i (0, t_2(x)) \right\}.$$

Then, the following theorem gives a minimal spanning set for C.

Theorem 2 *Let C be an additive (a, b)-double polycyclic code over $\mathbb{F}_{p^2}$ of length $l+m$ where*

$$C = \langle (\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)), (t_1(x), \omega l_3(x) + l_4(x)), (0, \omega g_2(x) + u_2(x)), (0, t_2(x)) \rangle .$$

Then, $S = \bigcup_{i=1}^{4} S_i$ is a minimal spanning set for C and

$$|C| = |Ker(\varphi)| \, |\varphi(C)| = p^{l - \deg(g_1(x))} \cdot p^{l - \deg(t_1(x))} \cdot p^{m - \deg(g_2(x))} \cdot p^{m - \deg(t_2(x))}.$$

Proof Let $\mathbf{c}(x) \in C$. Then, for some polynomials $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$ and $\lambda_4(x) \in \mathbb{F}_p[x]$,

$$\mathbf{c}(x) = \lambda_1(x)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) + \lambda_2(x)(t_1(x), \omega l_3(x) + l_4(x)) + \lambda_3(x)(0, \omega g_2(x) + u_2(x)) + \lambda_4(x)(0, t_2(x)) .$$

If $\deg(\lambda_4(x)) < m - \deg(t_2(x))$, then $\lambda_4(x)(0, t_2(x)) \in \langle S_4 \rangle$. If not, then we have $\lambda_4(x) = k_1(x)\left(\frac{x^m - b(x)}{t_2(x)}\right) + r_1(x)$ where $r_1(x) = 0$ or $\deg(r_1(x)) < m - \deg(t_2(x))$. Therefore,

$$\begin{aligned}\lambda_4(x)(0, t_2(x)) &= \left(k_1(x)\left(\frac{x^m - b(x)}{t_2(x)}\right) + r_1(x)\right)(0, t_2(x)) \\ &= r_1(x)(0, t_2(x)) \in \langle S_4 \rangle .\end{aligned}$$

If $\deg(\lambda_3(x)) < m - \deg(g_2(x))$, then $\lambda_3(x)(0, \omega g_2(x) + u_2(x)) \in \langle S_3 \rangle$. Otherwise, we can write $\lambda_3(x) = k_2(x)\left(\frac{x^m - b(x)}{g_2(x)}\right) + r_2(x)$ where $r_2(x) = 0$ or $\deg(r_2(x)) < m - \deg(g_2(x))$. Then,

$$\begin{aligned}\lambda_3(x)(0, \omega g_2(x) + u_2(x)) &= \left(k_2(x)\left(\frac{x^m - b(x)}{g_2(x)}\right) + r_2(x)\right)(0, \omega g_2(x) + u_2(x)) \\ &= k_2(x)\left(\frac{x^m - b(x)}{g_2(x)}\right)(0, \omega g_2(x) + u_2(x)) \\ &\quad + r_2(x)(0, \omega g_2(x) + u_2(x)) .\end{aligned}$$

Then, $r_2(x)(0, \omega g_2(x) + u_2(x)) \in \langle S_3 \rangle$ and

$$k_2(x)\left(\frac{x^m - b(x)}{g_2(x)}\right)(0, \omega g_2(x) + u_2(x)) = k_2(x)\left(\frac{x^m - b(x)}{g_2(x)}\right)(0, u_2(x)) .$$

Recall that $t_2(x) \mid \left(\frac{x^m - b(x)}{g_2(x)}\right) u_2(x)$ and say $\left(\frac{x^m - b(x)}{g_2(x)}\right) u_2(x) = t_2(x)\eta_1(x)$. Then, we get

$$k_2(x)\left(\frac{x^m - b(x)}{g_2(x)}\right)(0, u_2(x)) = \eta_1(x)(0, t_2(x)) \in \langle S_4 \rangle ,$$

which implies that $\left(k_2(x)\left(\frac{x^m - b(x)}{g_2(x)}\right) + r_2(x)\right)(0, \omega g_2(x) + u_2(x)) \in \langle S_3, S_4 \rangle$.

If $\deg(\lambda_2(x)) < l - \deg(t_1(x))$, then $\lambda_2(x)(t_1(x), \omega l_3(x) + l_4(x)) \in \langle S_2 \rangle$. If not, we have $\lambda_2(x) = k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right) + r_3(x)$ where $r_3(x) = 0$ or $\deg(r_3(x)) < l - \deg(t_1(x))$. Then,

$$\begin{aligned}\lambda_2(x)(t_1(x), \omega l_3(x) + l_4(x)) &= \left(k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right) + r_3(x)\right). \\ &\quad (t_1(x), \omega l_3(x) + l_4(x)) \\ &= k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right)(t_1(x), \omega l_3(x) + l_4(x)) \\ &\quad + r_3(x)(t_1(x), \omega l_3(x) + l_4(x)) \\ &= k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right)(0, \omega l_3(x) + l_4(x)) \\ &\quad + r_3(x)(t_1(x), \omega l_3(x) + l_4(x)) .\end{aligned}$$

Then, $r_3(x)(t_1(x), \omega l_3(x) + l_4(x)) \in \langle S_2 \rangle$ and

$$k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right)(0, \omega l_3(x) + l_4(x)) \in Ker(\varphi).$$

In this case,

$$k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right)(\omega l_3(x) + l_4(x)) \in J = \langle \omega g_2(x) + u_2(x), t_2(x) \rangle,$$

which gives

$$k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right)(\omega l_3(x) + l_4(x)) = \eta_2(x)(\omega g_2(x) + u_2(x)) + \eta_3(x) t_2(x)$$

and so

$$\begin{aligned} k_3(x)\left(\frac{x^l - a(x)}{t_1(x)}\right)(0, \omega l_3(x) + l_4(x)) = {} & \eta_2(x)(0, \omega g_2(x) + u_2(x)) \\ & + \eta_3(x)(0, t_2(x)) \in \langle S_3, S_4 \rangle. \end{aligned}$$

If $\deg(\lambda_1(x)) < l - \deg(g_1(x))$, then $\lambda_1(x)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) \in \langle S_1 \rangle$. Otherwise, we have $\lambda_1(x) = k_4(x)\left(\frac{x^l - a(x)}{g_1(x)}\right) + r_4(x)$ where $r_4(x) = 0$ or $\deg(r_4(x)) < l - \deg(g_1(x))$. Then,

$$\begin{aligned} & \lambda_1(x)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) \\ & = \left(k_4(x)\left(\frac{x^l - a(x)}{g_1(x)}\right) + r_4(x)\right)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) \\ & = \left(k_4(x)\left(\frac{x^l - a(x)}{g_1(x)}\right)\right)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) \\ & \quad + r_4(x)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)). \end{aligned}$$

Note that $r_4(x)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) \in \langle S_1 \rangle$. Moreover,

$$\begin{aligned} & k_4(x)\left(\frac{x^l - a(x)}{g_1(x)}\right)(\omega g_1(x) + u_1(x), \omega l_1(x) + l_2(x)) \\ & = k_4(x)\left(\frac{x^l - a(x)}{g_1(x)}\right)(0, \omega l_1(x) + l_2(x)) \\ & \quad + k_4(x)\left(\frac{x^l - a(x)}{g_1(x)}\right)(u_1(x), \omega l_1(x) + l_2(x)). \end{aligned}$$

Note that $k_4(x)\left(\frac{x^l-a(x)}{g_1(x)}\right)(0,\omega l_1(x)+l_2(x))\in\langle S_3,S_4\rangle$. Since

$$t_1(x)\mid\left(\frac{x^l-a(x)}{g_1(x)}\right)u_1(x),$$

say $t_1(x)\mid\left(\frac{x^l-a(x)}{g_1(x)}\right)u_1(x)=t_1(x)\eta_4(x)$. Then, we have

$$\begin{aligned}\left(\frac{x^l-a(x)}{g_1(x)}\right)k_4(x)&=(u_1(x),\omega l_1(x)+l_2(x))\\&=k_4(x)\left(t_1(x)\eta_4(x),\left(\frac{x^l-a(x)}{g_1(x)}\right)(\omega l_1(x)+l_2(x))\right).\end{aligned}$$

It follows from $(t_1(x),\omega l_3(x)+l_4(x))\in C$ that

$$\begin{aligned}&k_4(x)\left(t_1(x)\eta_4(x),\left(\frac{x^l-a(x)}{g_1(x)}\right)(\omega l_1(x)+l_2(x))\right)\\&\quad-k_4(x)\eta_4(x)(t_1(x),\omega l_3(x)+l_4(x))=(0,\omega z_1(x)+z_2(x))\in Ker(\varphi)\end{aligned}$$

for some $z_i(x)\in\mathbb{F}_p[x]$, and so $\omega z_1(x)+z_2(x)\in J=\langle\omega g_2(x)+u_2(x),t_2(x)\rangle$, which implies that

$$\omega z_1(x)+z_2(x)=\eta_5(x)(\omega g_2(x)+u_2(x))+\eta_6(x)t_2(x)$$

and

$$\begin{aligned}(0,\omega z_1(x)+z_2(x))&=\eta_5(x)(0,\omega g_2(x)+u_2(x))\\&\quad+\eta_6(x)(0,t_2(x))\in\langle S_3,S_4\rangle.\end{aligned}$$

Finally, we deduce from $k_4(x)\eta_4(x)(t_1(x),\omega l_3(x)+l_4(x))\in\langle S_2,S_3,S_4\rangle$ and the last equality that

$$k_4(x)\left(t_1(x)\eta_4(x),\left(\frac{x^l-a(x)}{g_1(x)}\right)(\omega l_1(x)+l_2(x))\right)\in\langle S_2,S_3,S_4\rangle,$$

which completes the proof. □

Example 1 Let $\mathbb{F}_4=\{0,1,\omega,\omega^2\}$, where $\omega^2=\omega+1$. For $x^l-a(x)=x+1$ and $x^m-b(x)=x^7+x^6+x^5+x^3+1$ over $\mathbb{F}_4$, let $g_1(x)=t_1(x)=1$, $u_1(x)=0$, $g_2(x)=x^2+x+1$, $t_2(x)=x^3+x^2+1$, $u_2(x)=x$, $l_1(x)=x$, $l_2(x)=x^2$, $l_3(x)=1+x$ and $l_4(x)=1$. Take C as an additive (a,b)-double polycyclic code over $\mathbb{F}_4$ of length $1+7$ generated by above polynomials over $\mathbb{F}_4$. Then, by Theorem 2, C is generated by the matrix

$$\begin{pmatrix} \omega & 0 & \omega & 1 & 0 & 0 & 0 & 0 \\ 1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & \omega^2 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & \omega^2 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & \omega^2 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & \omega^2 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & \omega^2 & \omega \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{11\times 8}$$

and C is an $\left(8, 2^{11}, 3\right)_4$ additive code that contains twice as many codewords as optimal linear code $[8, 5, 3]_4$ based on [21].

Note that if either $l = 0$ or $m = 0$ for an additive (a, b)-double polycyclic codes over $\mathbb{F}_{p^2}$ of length $n = l + m$, then it reduces an additive polycyclic code over $\mathbb{F}_{p^2}$ induced by either b or a, respectively. Let $\mathbb{F}_9 = \mathbb{F}_3(\omega)$ where $\omega^2 = \omega + 1$. Now, we give some examples of additive polycyclic codes over $\mathbb{F}_9$ in Table 1 which contain triple as many codewords as optimal linear code of same length and minimum distance over $\mathbb{F}_9$ based on [21].

For a vector $(d|t) = (d_0, \ldots, d_{l-1}|\, t_0, \ldots, t_{m-1}) \in \mathbb{F}_{p^2}^{l+m}$, write

$$(d|t) = \left(d_{0,1} + \omega d_{0,2}, \ldots, d_{l-1,1} + \omega d_{l-1,2} \middle|\, t_{0,1} + \omega t_{0,2}, \ldots, t_{m-1,1} + \omega t_{m-1,2}\right),$$

and define the maps $\theta_1 : \mathbb{F}_{p^2}^{l+m} \to \mathbb{F}_p^{2(l+m)}$ by

$$\begin{aligned} \theta_1(d|t) = \big(&d_{0,1} + d_{0,2}, \ldots, d_{l-1,1} + d_{l-1,2}, d_{1,2}, \ldots, d_{l-1,2}, \\ &t_{0,1} + t_{0,2}, \ldots, t_{m-1,1} + t_{m-1,2}, t_{1,2}, \ldots, t_{m-1,2}\big), \end{aligned}$$

and $\theta_2, \theta_3 : \mathbb{F}_{p^2}^{l+m} \to \mathbb{F}_p^{(l+m)}$ by

$$\theta_2(d|t) = \left(\omega d_0 + \omega^p d_0^p, \ldots, \omega d_{l-1} + \omega^p d_{l-1}^p, \omega t_0 + \omega^p t_0^p, \ldots, \omega t_{m-1} + \omega^p t_{m-1}^p\right),$$

$$\theta_3(d|t) = \left(Tr(d_0), \ldots, Tr(d_{l-1}), Tr(t_0), \ldots, Tr(t_{m-1})\right),$$

where Tr is trace map from $\mathbb{F}_{p^2}$ to $\mathbb{F}_p$.

We mean by BKLC the best known linear code for given length and dimension based on [21]. In Table 2, we present some optimal linear codes and BKLCs over $\mathbb{F}_3$ as $\theta_1, \theta_2, \theta_3$ images of additive polycyclic codes over $\mathbb{F}_9$ based on [21]. The codes with parameters $[n, k, d]_3^*$ means BKLCs.

Table 1 Additive polycyclic codes over $\mathbb{F}_9$ that contain triple as many codewords as optimal linear code of same length and minimum distance over $\mathbb{F}_9$ for lengths $10 < n < 20$

$\left(n, 3 \cdot 9^k, d\right)_9$	$[n, k, d]_9$	$\langle \omega g_1(x) + u_1(x), t_1(x) \rangle$	$x^l - a(x)$
$\left(11, 3 \cdot 9^5, 6\right)_9$	$[11, 5, 6]_9$	$\omega\left(x^5 + 2x^3 + x^2 + 2x + 2\right) + x^5 + 2x^4 + x^3, x^6 + x^4 + 2x^3 + 2x^2 + 2x + 1$	$x^{11} + 2$
$\left(11, 3 \cdot 9^8, 3\right)_9$	$[11, 8, 3]_9$	$\omega(x+1) + 2x^3 + x^2 + x + 2, x^4 + 2x^2 + 2x + 1$	$x^{11} + x^{10} + 2x^9 + 2$
$\left(12, 3 \cdot 9^9, 3\right)_9$	$[12, 9, 3]_9$	$\omega(x+2) + 2x^3 + x^2 + x, x^4 + 2x^2 + x + 1$	$x^{12} + x^{10} + x^9 + x^8 + 2x^7 + x^6 + 2$
$\left(13, 3 \cdot 9^{10}, 3\right)_9$	$[13, 10, 3]_9$	$\omega(x+2) + 2x^3 + x, x^4 + x^3 + x^2 + 1$	$x^{13} + x^{12} + 2x^7 + x^5 + 2x^4 + 2x^3 + 2x^2 + 1$
$\left(14, 3 \cdot 9^{11}, 3\right)_9$	$[14, 11, 3]_9$	$\omega(x+2) + 2x^3 + x^2, x^4 + x^3 + x^2 + x + 2$	$x^{14} + 2x^{12} + x^{11} + x^{10} + x^9 + x^8 + 2$
$\left(15, 3 \cdot 9^{12}, 3\right)_9$	$[15, 12, 3]_9$	$\omega(x+2) + 2x^2 + 2, x^4 + x^3 + 2x^2 + 2x + 2$	$x^{15} + 2x^{13} + 2x^{12} + x^{10} + 2x^9 + 2x^7 + 2$
$\left(16, 3 \cdot 9^{13}, 3\right)_9$	$[16, 13, 3]_9$	$\omega(x+1) + 2x^3 + x^2 + x, x^4 + x^3 + x^2 + 1$	$x^{16} + 2x^{15} + x^{13} + x^{12} + x^{11} + 2x^9 + 2x^8 + 2$
$\left(17, 3 \cdot 9^{14}, 3\right)_9$	$[17, 14, 3]_9$	$\omega(x+2) + x^3 + 2x^2 + 2x + 2, x^4 + 2x^3 + 2x^2 + x + 2$	$x^{17} + 2x^{15} + x^{14} + 2$
$\left(18, 3 \cdot 9^{15}, 3\right)_9$	$[18, 15, 3]_9$	$\omega(x+2) + x^2 + x, x^4 + x^3 + x^2 + 1$	$x^{18} + 2x^{14} + x^{12} + 2$
$\left(19, 3 \cdot 9^{16}, 3\right)_9$	$[19, 16, 3]_9$	$\omega(x+2) + x^2 + 2x + 2, x^4 + x^3 + 2x^2 + 2x + 2$	$x^{19} + x^{18} + x^{16} + x^{15} + 2x^{14} + x^{13} + 2$

Table 2 Optimal ternary linear codes and BKLCs derived from additive polycyclic codes over $\mathbb{F}_9$ via the maps θ_1, θ_2 and θ_3

$[n, k, d]_3$	Map	$\langle \omega g_1(x) + u_1(x), t_1(x) \rangle$	$x^l - a(x)$
$[7, 3, 4]_3$	θ_3	$\omega(x+2) + x + 2, x^4 + x^2 + x + 1$	$x^7 + 2x^6 + 2x^5 + 2x^4 + x^3 + 2x^2 + 2$
$[8, 4, 4]_3$	θ_3	$\omega(x+2) + x + 2, x^4 + 2x^3 + x + 1$	$x^8 + 2x^7 + 2x^6 + 2x^4 + 2$
$[9, 4, 5]_3$	θ_3	$\omega(x+1) + x + 1, x^5 + 2x^3 + x^2 + 2x + 2$	$x^9 + 2x^8 + x^7 + x^6 + 2$
$[10, 6, 4]_3$	θ_3	$\omega\left(x^5 + x^4 + 2x + 1\right) + x^2 + x + 1, x^4 + x^3 + 2x + 1$	$x^{10} + x^9 + 2x^8 + x^7 + 2x^5 + 2$
$[10, 5, 5]_3$	θ_3	$\omega\left(x^2 + x + 2\right) + x^2 + x + 2, x^5 + 2x^3 + x^2 + 2x + 2$	$x^{10} + x^9 + x^8 + 2x^7 + 2x^5 + 2$
$[11, 8, 3]_3$	θ_3	$\omega\left(x^5 + x^4 + x^3 + x + 1\right) + x + 2, x^3 + x^2 + 2$	$x^{11} + x^{10} + 2x^9 + 2$
$[11, 3, 7]_3$	θ_3	$\omega\left(x^3 + x^2 + x + 2\right) + x^3 + x^2 + x + 2, x^8 + 2x^7 + x^5 + x^4 + x^3 + 2x^2 + x + 1$	$x^{11} + 2x^8 + 2$
$[14, 10, 3]_3$	θ_1	$\omega(x+2) + x + 2, x^3 + x + 2$	$x^7 + 2x^6 + 2x^5 + 2x^4 + 2$
$[14, 3, 9]_3$	θ_1	$\omega\left(x^4 + 2x^3 + 2x^2 + 1\right) + x^4 + 2x^3 + 2x^2 + 1, x^7 + 2x^5 + x^3 + 2x^2 + x + 2$	$x^7 + 2x^5 + x^3 + 2x^2 + x + 2$
$[16, 12, 3]_3$	θ_1	$\omega(x+2) + x + 2, x^3 + x + 2$	$x^8 + 2x^7 + x^6 + 2x^5 + 2x^4 + 2x^3 + 2$
$[16, 6, 7]_3$	θ_1	$\omega\left(x^3 + x^2 + x + 1\right) + x^3 + x^2 + x + 1, x^7 + x^6 + 2x^5 + 2x^4 + x^3 + 2$	$x^8 + 2x^7 + x^5 + x^3 + 2x + 2$
$[17, 12, 4]_3$	θ_2	$\omega\left(x^5 + x^4 + x^3 + x + 1\right) + x^5 + x^4 + x^3 + x + 1, x^8 + 2x^7 + x^6 + x^5 + x^4 + 1$	$x^{17} + 2x^{16} + 2$
$[18, 14, 3]_3$	θ_1	$\omega(x+1) + x + 1, x^3 + 2x^2 + x + 1$	$x^9 + x^8 + 2x^7 + x^6 + 2x^5 + x^4 + 2$

4 Duals of additive double polycyclic codes over $\mathbb{F}_{p^2}$

In this subsection, we are going to investigate the duals of additive (a, b)-double polycyclic code over $\mathbb{F}_{p^2}$ of length $l + m$ by letting $a(x) = \lambda$ and $b(x) = \beta$ for nonzero $\lambda, \beta \in \mathbb{F}_p$ and from now on we are going to call such a code as additive (λ, β)-double constacyclic code.

Let C be an additive (λ, β)-double constacyclic code over $\mathbb{F}_{p^2}$ of length $n = l + m$. Define $\mathbf{u}$ and $\mathbf{v}$ as two vectors $\mathbf{u} = \left(u_0, \ldots, u_{l-1} | u'_0 \ldots, u'_{m-1}\right)$ and $\mathbf{v} =$

Table 2 continued

$[n, k, d]_3$	Map	$\langle \omega g_1(x) + u_1(x), t_1(x) \rangle$	$x^l - a(x)$
$[20, 16, 3]_3$	θ_1	$\omega(x+1) + x + 1, x^3 + x^2 + x + 2$	$x^{10} + 2x^9 + x^8 + x^7 + 2x^5 + 2x^2 + x + 2$
$[22, 16, 4]_3$	θ_2	$\omega\left(x^4 + 2x^3 + 2x^2 + x + 2\right)$, $x^6 + 2x^5 + 2x^2 + 1$	$x^{22} + x^{21} + 2x^{20} + 2$
$[23, 19, 3]_3$	θ_3	$\omega(x+2) + x + 2, x^4 + x + 2$	$x^{23} + x^{22} + 2x^{21} + 2$
$[28, 22, 4]_3$	θ_2	$\omega(x+2), x^6 + 2x^4 + x + 1$	$x^{28} + x^{27} + 2x^{25} + 2$
$[31, 20, 6]_3$	θ_2	$\omega\left(x^{11} + 2x^{10} + 2x^9 + 2x^8 + x^4 + x^3 + 2x^2 + x + 2\right)$, $x^{11} + 2x^{10} + 2x^9 + 2x^8 + x^4 + x^3 + 2x^2 + x + 2$	$x^{31} + 2x^{30} + 2$
$[32, 25, 4]_3^*$	θ_1	$\omega\left(x^3 + 2x + 1\right) + x^3 + 2x + 1, x^4 + 2x^3 + x^2 + 1$	$x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{10} + 2x^8 + x^6 + 2$
$[34, 29, 3]_3^*$	θ_1	$\omega\left(x^2 + 2\right) + x^2 + 2, x^3 + 2x + 2$	$x^{17} + x^{16} + x^{14} + 2x^{13} + 2x^{12} + 2$
$[40, 35, 3]_3^*$	θ_1	$\omega(x+1) + x + 1, x^4 + x^3 + 1$	$x^{20} + x^{19} + x^{18} + 2x^{17} + 2x^{16} + 2x^{14} + x^{10} + 2$
$[44, 31, 6]_3^*$	θ_1	$\omega\left(x^3 + x + 1\right) + 2x^7 + 2x^5 + 2x^3 + 1, x^{10} + x^9 + 2x^8 + 2x^6 + 2x^2 + 2$	$x^{22} + 2x^{21} + 2x^{20} + 2x^{19} + 2$
$[44, 36, 4]_3^*$	θ_1	$\omega(x+1) + 2x^5 + x^4 + 2x, x^7 + 2x^6 + x^5 + x^3 + 2x + 1$	$x^{22} + x^{21} + x^{20} + 2$

$\left(v_0, \ldots, v_{l-1} | v'_0 \ldots, v'_{m-1}\right)$ in $\mathbb{F}_{p^2}^l \times \mathbb{F}_{p^2}^m$. We define Euclidean dual $C^\perp$ and Hermitian dual $C^{\perp_h}$ of the code C as usual way:

$$C^\perp = \left\{ \mathbf{v} \in \mathbb{F}_{p^2}^l \times \mathbb{F}_{p^2}^m : \langle \mathbf{u}, \mathbf{v} \rangle = 0, \ \forall \mathbf{u} \in C \right\},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=0}^{l-1} u_i v_i + \sum_{i=0}^{m-1} u'_i v'_i$ and

$$C^{\perp_h} = \left\{ \mathbf{v} \in \mathbb{F}_{p^2}^l \times \mathbb{F}_{p^2}^m : \langle \mathbf{u}, \mathbf{v} \rangle_h = 0, \ \forall \mathbf{u} \in C \right\},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle_h = \sum_{i=0}^{l-1} u_i v_i^p + \sum_{i=0}^{m-1} u'_i v_i'^p$.

For a vector $\mathbf{u} = \left(u_0, \ldots, u_{l-1} | \, u'_0 \ldots, u'_{m-1}\right) \in \mathbb{F}_{p^2}^l \times \mathbb{F}_{p^2}^m$, define $T_{\lambda,\beta}$ as

$$T_{\lambda,\beta}(\mathbf{u}) = \left(\lambda u_{l-1}, u_0, \ldots, u_{l-2} | \, \beta u'_{m-1}, u'_0 \ldots, u'_{m-2}\right)$$

and

$$T^i_{\lambda,\beta}(\mathbf{u}) = T^{i-1}_{\lambda,\beta}\left(T_{\lambda,\beta}(\mathbf{u})\right).$$

Theorem 3 *The Euclidean dual $C^{\perp}$ and the Hermitian dual $C^{\perp_h}$ of an additive (λ, β)-double constacyclic code C over $\mathbb{F}_{p^2}$ of length $l+m$ are additive $\left(\lambda^{-1}, \beta^{-1}\right)$-double constacyclic code over $\mathbb{F}_{p^2}$ of length $l+m$.*

Proof Let $\mathbf{v} = \left(v_0, \ldots, v_{l-1} | \, v'_0, \ldots, v'_{m-1}\right) \in C^{\perp}$. It is sufficient to show that $T_{\lambda^{-1},\beta^{-1}}(\mathbf{v}) = \left(\lambda^{-1} v_{l-1}, v_0, \ldots, v_{l-2} \middle| \, \beta^{-1} v'_{m-1}, v'_0 \ldots, v'_{m-2}\right) \in C^{\perp}$ to prove that the Euclidean dual $C^{\perp}$ is an additive $\left(\lambda^{-1}, \beta^{-1}\right)$-double constacyclic code. For any vector $\mathbf{u} = \left(u_0, \ldots, u_{l-1} | \, u'_0 \ldots, u'_{m-1}\right) \in C$, see that $T^d_{\lambda,\beta}(\mathbf{u}) = \mathbf{u}$ and

$$T^{d-1}_{\lambda,\beta}(\mathbf{u}) = \left(u_1, \ldots, u_{l-1}, \lambda^{-1} u_0 \middle| \, u'_1, \ldots, u'_{m-1}, \beta^{-1} u'_0\right) \in C,$$

where $d = (p-1)\, lcm\,(l, m)$. In this case, we get

$$\begin{aligned}
0 = \left\langle T^{d-1}_{\lambda,\beta}(\mathbf{u}), \mathbf{v} \right\rangle &= u_1 v_0 + u_2 v_1 + \cdots + u_{l-1} v_{l-2} + \lambda^{-1} u_0 v_{l-1} \\
&+ u'_1 v'_0 + u'_2 v'_1 + \ldots + u'_{m-1} v'_{m-2} + \beta^{-1} u'_0 v'_{m-1} \\
&= u_0 \lambda^{-1} v_{l-1} + u_1 v_0 + u_2 v_1 + \cdots + u_{l-1} v_{l-2} \\
&+ u'_0 \beta^{-1} v'_{m-1} + u'_1 v'_0 + u'_2 v'_1 + \ldots + u'_{m-1} v'_{m-2} \\
&= \left\langle \mathbf{u}, T_{\lambda^{-1},\beta^{-1}}(\mathbf{v}) \right\rangle,
\end{aligned}$$

which proves that $T_{\lambda^{-1},\beta^{-1}}(\mathbf{v}) \in C^{\perp}$.

For the Hermitian case, it follows from $\lambda^{-p} = \lambda^{-1}$, $\beta^{-p} = \beta^{-1}$ for nonzero $\lambda, \beta \in \mathbb{F}_p$ and $\mathbf{v} \in C^{\perp} \Leftrightarrow \mathbf{v}^p = \left(v_0^p, \ldots, v_{l-1}^p \middle| \, v_0'^p, \ldots, v_{m-1}'^p\right) \in C^{\perp_h}$ that $T_{\lambda^{-1},\beta^{-1}}(\mathbf{v}) \in C^{\perp}$ if and only if

$$\begin{aligned}
\left(T_{\lambda^{-1},\beta^{-1}}(\mathbf{v})\right)^p &= \left(\lambda^{-p} v_0^p, \ldots, v_{l-1}^p \middle| \, \beta^{-p} v_0'^p, \ldots, v_{m-1}'^p\right) \\
&= T_{\lambda^{-p},\beta^{-p}}\left(\mathbf{v}^p\right) \\
&= T_{\lambda^{-1},\beta^{-1}}\left(\mathbf{v}^p\right) \in C^{\perp_h},
\end{aligned}$$

which completes the proof. □

Before exploring the relation between the orthogonality of two vectors and the product of their polynomial representations, we give an example to clarify the relation by letting $l = 2, m = 3$ and $p = 5$.

Let $\mathbf{u} = \left(u_0, u_1 | u_0', u_1', u_2'\right)$ and $\mathbf{v} = \left(v_0, v_1 | v_0', v_1', v_2'\right)$ be two vectors over $\mathbb{F}_{25}$. Note that $o(\lambda) = o(\beta) = 4$ for nonzero $\lambda, \beta \in \mathbb{F}_5$, and there are exactly $d = (p-1)\, lcm(l, m) = 24$ different shifts of the vector $\mathbf{u}$. In this case, we have the following identities:

$$
\begin{aligned}
\langle \mathbf{v}, \mathbf{u} \rangle &= \underbrace{v_0u_0 + v_1u_1}_{\gamma_0} + \underbrace{v_0'u_0' + v_1'u_1' + v_2'u_2'}_{\sigma_0} = \gamma_0 + \sigma_0,\\
\left\langle \mathbf{v}, T_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \underbrace{\lambda v_0u_1 + v_1u_0}_{\gamma_1} + \underbrace{\beta v_0'u_2' + v_1'u_0' + v_2'u_1'}_{\sigma_1} = \gamma_1 + \sigma_1,\\
\left\langle \mathbf{v}, T^2_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda v_0u_0 + \lambda v_1u_1 + \underbrace{\beta v_0'u_1' + \beta v_1'u_2' + v_2'u_0'}_{\sigma_2} = \lambda\gamma_0 + \sigma_2,\\
\left\langle \mathbf{v}, T^3_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^2 v_0u_1 + \lambda v_1u_0 + \beta v_0'u_0' + \beta v_1'u_1' + \beta v_2'u_2' = \lambda\gamma_1 + \beta\sigma_0,\\
\left\langle \mathbf{v}, T^4_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^2 v_0u_0 + \lambda^2 v_1u_1 + \beta^2 v_0'u_2' + \beta v_1'u_0' + \beta v_2'u_1' = \lambda^2\gamma_0 + \beta\sigma_1\\
\left\langle \mathbf{v}, T^5_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^3 v_0u_1 + \lambda^2 v_1u_0 + \beta^2 v_0'u_1' + \beta^2 v_1'u_2' + \beta v_2'u_0' = \lambda^2\gamma_1 + \beta\sigma_2,\\
\left\langle \mathbf{v}, T^6_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^3 v_0u_0 + \lambda^3 v_1u_1 + \beta^2 v_0'u_0' + \beta^2 v_1'u_1' + \beta^2 v_2'u_2' = \lambda^3\gamma_0 + \beta^2\sigma_0,\\
\left\langle \mathbf{v}, T^7_{\lambda,\beta}(\mathbf{u}) \right\rangle &= v_0u_1 + \lambda^3 v_1u_0 + \beta^3 v_0'u_2' + \beta^2 v_1'u_0' + \beta^2 v_2'u_1' = \lambda^3\gamma_1 + \beta^2\sigma_1,\\
\left\langle \mathbf{v}, T^8_{\lambda,\beta}(\mathbf{u}) \right\rangle &= v_0u_0 + v_1u_1 + \beta^3 v_0'u_1' + \beta^3 v_1'u_2' + \beta^2 v_2'u_0' = \gamma_0 + \beta^2\sigma_2,\\
\left\langle \mathbf{v}, T^9_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda v_0u_1 + v_1u_0 + \beta^3 v_0'u_0' + \beta^3 v_1'u_1' + \beta^3 v_2'u_2' = \gamma_1 + \beta^3\sigma_0,\\
\left\langle \mathbf{v}, T^{10}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda v_0u_0 + \lambda v_1u_1 + v_0'u_2' + \beta^3 v_1'u_0' + \beta^3 v_2'u_1' = \lambda\gamma_0 + \beta^3\sigma_1,\\
\left\langle \mathbf{v}, T^{11}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^2 v_0u_1 + \lambda v_1u_0 + v_0'u_1' + v_1'u_2' + \beta^3 v_2'u_0' = \lambda\gamma_1 + \beta^3\sigma_2,\\
\left\langle \mathbf{v}, T^{12}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^2 v_0u_0 + \lambda^2 v_1u_1 + v_0'u_0' + v_1'u_1' + v_2'u_2' = \lambda^2\gamma_0 + \sigma_0,\\
\left\langle \mathbf{v}, T^{13}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^3 v_0u_1 + \lambda^2 v_1u_0 + \beta v_0'u_2' + v_1'u_0' + v_2'u_1' = \lambda^2\gamma_1 + \sigma_1,\\
\left\langle \mathbf{v}, T^{14}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda^3 v_0u_0 + \lambda^3 v_1u_1 + \beta v_0'u_1' + \beta v_1'u_2' + v_2'u_0' = \lambda^3\gamma_0 + \sigma_2,\\
\left\langle \mathbf{v}, T^{15}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= v_0u_1 + \lambda^3 v_1u_0 + \beta v_0'u_0' + \beta v_1'u_1' + \beta v_2'u_2' = \lambda^3\gamma_1 + \beta\sigma_0,\\
\left\langle \mathbf{v}, T^{16}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= v_0u_0 + v_1u_1 + \beta^2 v_0'u_2' + \beta v_1'u_0' + \beta v_2'u_1' = \gamma_0 + \beta\sigma_1,\\
\left\langle \mathbf{v}, T^{17}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda v_0u_1 + v_1u_0 + \beta^2 v_0'u_1' + \beta^2 v_1'u_2' + \beta v_2'u_0' = \gamma_1 + \beta\sigma_2\\
\left\langle \mathbf{v}, T^{18}_{\lambda,\beta}(\mathbf{u}) \right\rangle &= \lambda v_0u_0 + \lambda v_1u_1 + \beta^2 v_0'u_0' + \beta^2 v_1'u_1' + \beta^2 v_2'u_2' = \lambda\gamma_0 + \beta^2\sigma_0,
\end{aligned}
$$

$$\begin{aligned}
\left\langle \mathbf{v}, T_{\lambda,\beta}^{19}(\mathbf{u}) \right\rangle &= \lambda^2 v_0 u_1 + \lambda v_1 u_0 + \beta^3 v_0' u_2' + \beta^2 v_1' u_0' + \beta^2 v_2' u_1' = \lambda\gamma_1 + \beta^2\sigma_1, \\
\left\langle \mathbf{v}, T_{\lambda,\beta}^{20}(\mathbf{u}) \right\rangle &= \lambda^2 v_0 u_0 + \lambda^2 v_1 u_1 + \beta^3 v_0' u_1' + \beta^3 v_1' u_2' + \beta^2 v_2' u_0' = \lambda^2\gamma_0 + \beta^2\sigma_2, \\
\left\langle \mathbf{v}, T_{\lambda,\beta}^{21}(\mathbf{u}) \right\rangle &= \lambda^3 v_0 u_1 + \lambda^2 v_1 u_0 + \beta^3 v_0' u_0' + \beta^3 v_1' u_1' + \beta^3 v_2' u_2' = \lambda^2\gamma_1 + \beta^3\sigma_0, \\
\left\langle \mathbf{v}, T_{\lambda,\beta}^{22}(\mathbf{u}) \right\rangle &= \lambda^3 v_0 u_0 + \lambda^3 v_1 u_1 + v_0' u_2' + \beta^3 v_1' u_0' + \beta^3 v_2' u_1' = \lambda^3\gamma_0 + \beta^3\sigma_1, \\
\left\langle \mathbf{v}, T_{\lambda,\beta}^{23}(\mathbf{u}) \right\rangle &= v_0 u_1 + \lambda^3 v_1 u_0 + v_0' u_1' + v_1' u_2' + \beta^3 v_2' u_0' = \lambda^3\gamma_1 + \beta^3\sigma_2.
\end{aligned}$$

Let $u(x) = u_0 + u_1 x$, $v^*(x) = v_1 + v_0 x$, $u'(x) = u_0' + u_1' x + u_1' x^2$ and $v'^*(x) = v_2' + v_1' x + v_0' x^2$. See that $u(x)\, v^*(x) = \gamma_1 + \gamma_0 x \pmod{\left(x^2 - \lambda\right)}$ and $u'(x)\, v'^*(x) = \sigma_2 + \sigma_1 x + \sigma_0 x^2 \pmod{\left(x^3 - \beta\right)}$. Now, construct the polynomial

$$\begin{aligned}
G(x) &= \left(\lambda^3\gamma_1 + \beta^3\sigma_2\right) + \left(\lambda^3\gamma_0 + \beta^3\sigma_1\right)x + \left(\lambda^2\gamma_1 + \beta^3\sigma_0\right)x^2 \\
&\quad + \left(\lambda^2\gamma_0 + \beta^2\sigma_2\right)x^3 + \left(\lambda\gamma_1 + \beta^2\sigma_1\right)x^4 + \left(\lambda\gamma_0 + \beta^2\sigma_0\right)x^5 \\
&\quad + \left(\gamma_1 + \beta\sigma_2\right)x^6 + \left(\gamma_0 + \beta\sigma_1\right)x^7 + \left(\lambda^3\gamma_1 + \beta\sigma_0\right)x^8 \\
&\quad + \left(\lambda^3\gamma_0 + \sigma_2\right)x^9 + \left(\lambda^2\gamma_1 + \sigma_1\right)x^{10} + \left(\lambda^2\gamma_0 + \sigma_0\right)x^{11} \\
&\quad + \left(\lambda\gamma_1 + \beta^3\sigma_2\right)x^{12} + \left(\lambda\gamma_0 + \beta^3\sigma_1\right)x^{13} + \left(\gamma_1 + \beta^3\sigma_0\right)x^{14} \\
&\quad + \left(\gamma_0 + \beta^2\sigma_2\right)x^{15} + \left(\lambda^3\gamma_1 + \beta^2\sigma_1\right)x^{16} + \left(\lambda^3\gamma_0 + \beta^2\sigma_0\right)x^{17} \\
&\quad + \left(\lambda^2\gamma_1 + \beta\sigma_2\right)x^{18} + \left(\lambda^2\gamma_0 + \beta\sigma_1\right)x^{19} + \left(\lambda\gamma_1 + \beta\sigma_0\right)x^{20} \\
&\quad + \left(\lambda\gamma_0 + \sigma_2\right)x^{21} + \left(\gamma_1 + \sigma_1\right)x^{22} + \left(\gamma_0 + \sigma_0\right)x^{23} \\
&= \left(\gamma_1 + \gamma_0 x\right)\left(\sum_{i=0}^{11}\lambda^i x^{22-2i}\right) + \left(\sigma_2 + \sigma_1 x + \sigma_0 x^2\right)\left(\sum_{j=0}^{7}\beta^j x^{21-3j}\right) \\
&= \left[u(x)\, v^*(x) \bmod \left(x^2 - \lambda\right)\right]\left(\frac{x^{24}-1}{x^2-\lambda}\right) \\
&\quad + \left[u'(x)\, v'^*(x) \bmod \left(x^3 - \beta\right)\right]\left(\frac{x^{24}-1}{x^3-\beta}\right),
\end{aligned}$$

where $p = 5$, $l = 2$, $m = 3$ and $d = 24 = (p-1)\, lcm(l, m)$. Then, we can say that $\mathbf{v}$ is orthogonal to all shifts of the vector $\mathbf{u}$ under $T_{\lambda,\beta}$ if and only if the polynomial $G(x) = 0$. In general, $\left\langle \mathbf{v}, T_{\lambda,\beta}^{i}(\mathbf{u}) \right\rangle$ and the polynomial $G(x)$ can be given as following:

Let $\mathbf{u} = \left(u_0, \ldots, u_{l-1} \mid u_0', \ldots, u_{m-1}'\right)$, $\mathbf{v} = \left(v_0, \ldots, v_{l-1} \mid v_0', \ldots, u_{m-1}'\right)$ be two vectors over $\mathbb{F}_{p^2}$. For $0 \le i \le d-1$, set $i = i_1 l + j_1 = i_2 m + j_2$ where $0 \le j_1 < l$ and $0 \le j_2 < m$.

If $j_1 = j_2 = 0$, then

$$\left\langle \mathbf{v}, T^i_{\lambda,\beta}(\mathbf{u}) \right\rangle = \lambda^{i_1} \sum_{j=0}^{l-1} u_j v_j + \beta^{i_2} \sum_{j=0}^{m-1} u'_j v'_j.$$

If $j_1 \neq 0$ and $j_2 = 0$, then

$$\left\langle \mathbf{v}, T^i_{\lambda,\beta}(\mathbf{u}) \right\rangle = \lambda^{i_1+1} \sum_{j=0}^{j_1-1} u_{j-j_1} v_j + \lambda^{i_1} \sum_{j=j_1}^{l-1} u_{j-j_1} v_j + \beta^{i_2} \sum_{j=0}^{m-1} u'_j v'_j.$$

If $j_1 = 0$ and $j_2 \neq 0$, then

$$\left\langle \mathbf{v}, T^i_{\lambda,\beta}(\mathbf{u}) \right\rangle = \lambda^{i_1} \sum_{j=0}^{l-1} u_j v_j + \beta^{i_2+1} \sum_{j=0}^{j_2-1} u'_{j-j_2} v'_j + \beta^{i_2} \sum_{j=j_2}^{m-1} u'_{j-j_2} v'_j.$$

If $j_1 \neq 0$ and $j_2 \neq 0$, then

$$\begin{aligned}\left\langle \mathbf{v}, T^i_{\lambda,\beta}(\mathbf{u}) \right\rangle = {} & \lambda^{i_1+1} \sum_{j=0}^{j_1-1} u_{j-j_1} v_j + \lambda^{i_1} \sum_{j=j_1}^{l-1} u_{j-j_1} v_j \\ & + \beta^{i_2+1} \sum_{j=0}^{j_2-1} u'_{j-j_2} v'_j + \beta^{i_2} \sum_{j=j_2}^{m-1} u'_{j-j_2} v'_j.\end{aligned}$$

Now, for $0 \leq i \leq d-1$ define S_i to be $S_i = \left\langle \mathbf{v}, T^i_{\lambda,\beta}(\mathbf{u}) \right\rangle$ and the polynomial $G(x)$ as $G(x) = \sum_{i=0}^{d-1} S_{d-1-i} x^i$. Then, we give the following without proof.

Theorem 4 *Let* $\boldsymbol{u} = \left(u_0, \ldots, u_{l-1} \mid u'_0, \ldots, u'_{m-1}\right), \boldsymbol{v} = \left(v_0, \ldots, v_{l-1} \mid v'_0, \ldots, u'_{m-1}\right)$ *be two vectors over* $\mathbb{F}_{p^2}$ *and let the polynomial* $G(x)$ *be defined as above. Set* $d = (p-1)\, lcm(l, m)$. *Then,* $\boldsymbol{v}$ *is (Euclidean) orthogonal to all shifts of the vector* $\boldsymbol{u}$ *under* $T_{\lambda,\beta}$ *if and only if* $S_i = \left\langle \boldsymbol{v}, T^i_{\lambda,\beta}(\boldsymbol{u}) \right\rangle = 0$ *for* $0 \leq i \leq d-1$, *i.e., the polynomial* $G(x) = 0$.

5 Quantum codes from additive double polycyclic codes over $\mathbb{F}_{p^2}$

In this subsection, we are going to do a survey for the parameters of quantum error correcting codes from additive polycyclic codes over $\mathbb{F}_{p^2}$. Let us first give a brief recall for quantum codes. An $((n, K, d))_q$ quantum code is a K-dimensional subspace of the q^n-dimensional complex vector space $\mathbb{C}^{\otimes q^n}$ where $\otimes$ denotes the tensor product. In particular, if $K = q^k$, then it is denoted by $[[n, k, d]]_q$ and such a quantum code encodes

Table 3 Ternary quantum codes derived from dual-containing linear codes over $\mathbb{F}_9$ that are obtained from additive polycyclic codes over $\mathbb{F}_9$ via the maps θ_2 and θ_3

$[[n, k, d]]_3$	Map	$\langle \omega g_1(x) + u_1(x), t_1(x) \rangle$	$x^l - a(x)$
$[[9, 3, 3]]_3^*$	θ_3	$\omega\left(x^4 + x^3 + 2x^2 + 2\right) + 2x^7 + x^6 + 2x^5 + 2x^4, x^8 + x^6 + x^3 + x^2 + x + 1$	$x^9 + 2x^8 + x^7 + 2x^6 + x^4 + 2$
$[[10, 2, 4]]_3^*$	θ_3	$\omega\left(x^2 + 2\right) + x^2 + 2, x^4 + 2x^3 + x + 1$	$x^{10} + x^9 + 2x^8 + 2x^6 + 2x^5 + 2x^4 + 2$
$[[11, 7, 2]]_3^*$	θ_3	$\omega\left(x^4 + 2x^3 + x^2 + 2x + 2\right) + 2x^8 + 2x^7 + x^5, x^9 + x^8 + x^6 + x^4 + x^2 + 1$	$x^{11} + x^{10} + 2x^9 + 2$
$[[11, 1, 5]]_3^*$	θ_2	$\omega\left(x^7 + 2x^6 + x^5 + x^3 + 2x^2 + 2\right) + x^7 + 2x^6 + 2x^5 + 2x^4 + x^2, x^9 + 2x^8 + 2x^7 + 2x^6 + x^5 + 2x^3 + 2x^2 + 1$	$x^{11} + 2x^{10} + x^9 + 2x^7 + x^6 + x^5 + 2x^4 + x^3 + 2x^2 + 2$
$[[12, 8, 2]]_3^*$	θ_2	$\omega\left(x^5 + x^3 + 2x^2 + 2x + 1\right) + 2x^5 + x^4, x^7 + x^6 + x^5 + x^3 + x^2 + 2x + 2$	$x^{12} + x^{11} + 2x^{10} + 2$
$[[12, 4, 3]]_3^{**}$	θ_2	$\omega\left(x^5 + 2x^4 + x^2 + 2x + 1\right) + 2x^{10} + 2x^9 + x^6 + x^5 + 2x^4 + x^2 + x, x^{11} + 2x^{10} + x^9 + 2x^8 + x^7 + 2x^4 + x + 1$	$x^{12} + x^{11} + 2x^{10} + x^9 + 2x^8 + 2x^7 + 2x^5 + x^4 + x^2 + 2$
$[[13, 7, 3]]_3^{**}$	θ_3	$\omega\left(x^9 + x^8 + x^7 + 2x^6 + x^4 + x^3 + 2\right) + 2x^2 + 2x + 2, x^3 + 2x^2 + 2x + 2$	$x^{13} + 2x^{12} + 2x^{11} + 2x^8 + 2$
$[[14, 6, 3]]_3^{**}$	θ_2	$\omega\left(x^5 + x^3 + x^2 + x + 1\right) + x^{12} + x^9 + 2x^8 + x^7 + x^6 + 2x^5 + x^2, x^{13} + 2x^{12} + 2x^{10} + 2x^9 + x^7 + 2x^6 + x^5 + 2x^4 + x^3 + 2x^2 + x + 2$	$x^{14} + 2x^{12} + 2x^{11} + x^{10} + 2x^9 + x^8 + 2$
$[[15, 7, 3]]_3^{**}$	θ_3	$\omega\left(2x^{14} + x^{13} + 2x^{11} + x^8 + 1\right) + x^7 + 2x^6 + 2x^5 + 2x + 2, 2x^{11} + x^9 + 2x^8 + 2x^7 + 2x^5 + x^4 + x^2$	$x^{15} + x^{14} + 2x^{13} + x^{11} + 2x^8 + 2$

k qudits to n qudits and corrects the errors up to $\left\lfloor \frac{d-1}{2} \right\rfloor$. The following construction, called CSS code construction, explores the analogue between linear codes over $\mathbb{F}_q$ and q-ary quantum codes.

Theorem 5 (Lemma 20, [2]) *Let C_i, $i = 1, 2$ be a linear code of parameters $[n, k_i, d_i]_q$ such that $C_2^{\perp} \leq C_1$. Then, there exists an $[[n, k_1 + k_2 - n, d]]_q$ quantum stabilizer code where $d = \min\left\{wt(c) : c \in \left(C_1 \backslash C_2^{\perp}\right) \cup \left(C_2 \backslash C_1^{\perp}\right)\right\}$. In particular,*

if C is an $[n, k, d]_q$ linear code containing its Euclidean dual, then there exists an $[[n, 2k - n, d]]_q$ quantum stabilizer code.

Let $\mathbb{F}_9 = \mathbb{F}_3(\omega)$ where $\omega^2 = \omega + 1$. Note that we say that an $[[n, k, d]]_3$ quantum code is almost optimal if an $[[n, k + 1, d]]_3$ or $[[n, k, d + 1]]_3$ quantum code is optimal based on [22], and an $[[n, k, d]]_3$ quantum code is the best known quantum code (BKQC) for length n and dimension k if an $[[n, k, d + 1]]_3$ quantum code doesn't exist based on [22]. As an application of additive polycyclic codes over $\mathbb{F}_9$, by taking θ_2 and θ_3 images of additive polycyclic codes over $\mathbb{F}_9$ we get dual-containing linear codes over $\mathbb{F}_3$, and then we obtain almost optimal and BKQCs for given length and dimension via CSS construction. We present them in Table 3 in which the quantum codes with "$*$" and "$**$" are BKQCs and almost optimal, respectively.

6 Conclusion

We define additive (a, b)-double polycyclic codes over $\mathbb{F}_{p^2}$ as a generalization of [18, 19] for two vectors $a, b \in \mathbb{F}_p^n$. We determine algebraic structure of these codes by giving their generator polynomials. We obtain additive polycyclic codes over $\mathbb{F}_9$ which have triple as many codewords as optimal linear code of same length and minimum distance over $\mathbb{F}_9$. We derive optimal linear codes over $\mathbb{F}_3$ from additive polycyclic codes over $\mathbb{F}_9$. We also prove that Euclidean dual and Hermitian dual of an additive (λ, β)-double constacyclic code over $\mathbb{F}_{p^2}$ are an additive $(\lambda^{-1}, \beta^{-1})$-double constacyclic code over $\mathbb{F}_{p^2}$. Finally, we deduce almost optimal and BKQCs over $\mathbb{F}_3$ from additive polycyclic codes over $\mathbb{F}_9$.

Acknowledgements We would like to thank the anonymous reviewers for their careful checking and valuable remarks that improved the presentation and the content of the paper. This study is supported by Yildiz Technical University Scientific Research Projects Coordination Department with Project Number FBA-2023-5424.

Declarations

Conflict of interest The authors declare that they have no conflict interest regarding the publication of this paper.

References

1. Calderbank, A.R., Rains, E.M., Shor, P.W., Sloane, N.J.A.: Quantum error correction via codes over $\mathbb{F}_4$. IEEE Trans. Inform. Theory. **44**, 1369–1387 (1998)
2. Ketkar, A., Klappenecker, A., Kumar, S., Sarvepalli, P.K.: Nonbinary stabilizer codes over finite fields. IEEE Trans. Inform. Theory. **52**, 4892–4914 (2006)
3. Aydoğdu, I., Siap, I.: The structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes: Bounds on the minimum distance. Appl. Math. Inf. Sci. **7**(6), 2271–2278 (2013)
4. Abualrub, T., Siap, I., Aydın, N.: $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. IEEE Trans. Inform. Theory. **60**(3), 1508–1513 (2014)
5. Aydoğdu, I., Siap, I.: On $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. Linear Multilinear Algebra. **63**(10), 2089–2102 (2015)

6. Aydoğdu, I., Abualrub, T., Siap, I.: On $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes. Int. J. Comput. Math. **92**(9), 1806–1814 (2015)
7. Borges, J., Fernandez-Córdoba, C., Ten-Valls, R.: $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, generator polynomials, and dual codes. IEEE Trans. Inform. Theory. **62**(11), 6348–6354 (2016)
8. Borges, J., Fernandez-Córdoba, C., Ten-Valls, R.: On $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes. Adv. Math. Commun. **12**(1), 169–179 (2018)
9. Shi, M., Wu, R., Solé, P.: Asymptotically good additive cyclic codes exist. IEEE Commun. Lett. **22**(10), 1980–1983 (2018)
10. Shi, M., Wu, R., Krotov, D.S.: On $Z_p Z_{p^k}$-additive codes and their duality. IEEE Trans. Inform. Theory. **65**(6), 3841–3847 (2019)
11. Aydoğdu, I., Gürsoy, F.: $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-cyclic codes. J. Appl. Math. Comput. **60**, 327–341 (2019)
12. Molaei, R., Khashyarmannesh, K.: $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}\mathbb{Z}_{p^t}$-additive cyclic codes. Adv. Math. Commun. (2022). https://doi.org/10.3934/amc.2022079
13. Shi, M., Li, S., Solé, P.: $\mathbb{Z}_2\mathbb{Z}_4$-additive quasi-cyclic cdoes. IEEE Trans. Inform. Theory. **67**(11), 7232–7239 (2021)
14. Shi, M., Xu, L., Solé, P.: Construction of isodual codes from polycirculant matrices. Des. Codes Cryptogr. **88**, 2547–2560 (2020)
15. Wu, R., Shi, M.: On $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes and their Gray images. Des. Codes Cryptogr. **90**, 2551–2562 (2022)
16. Wu, R., Shi, M., Solé, P.: On the structure of 1-generator quasi-polycyclic codes over finite chain rings. J. Appl. Math. Comput. **68**, 3491–3503 (2022)
17. Shi, M., Liu, N., Özbudak, F., Soléé, P.: Additive cyclic complementary dual codes over $\mathbb{F}_4$. Finite Fields Appl. **83**, 102087 (2022)
18. Karbaski, A.S., Abualrub, T., Aydın, N., Liu, P.: Additive polycyclic codes over $\mathbb{F}_4$ induced by binary vectors and some optimal codes. Adv. Math. Commun. (2022). https://doi.org/10.3934/amc.2022004
19. Sarı, M., Köroğlu, M.E.: Additive polycyclic codes over $\mathbb{F}_{p^2}$ for any prime p. In: Proceeding Book of 2nd International Conference on Engineering, Natural and Social Sciences (ICENSOS 2023), 4–6 April, Konya, Türkiye, 6–9 (2023)
20. Dinh, H.Q., Yadav, B.P., Pathak, S., Prasad, A., Upadhyay, A.K., Yamaka, W.: $\mathbb{F}_2[u]\mathbb{F}_2[u]$-additive cyclic codes are asymptotically good. J. Appl. Math. Commun. **69**, 1037–1056 (2023)
21. Grassl, M.: Bounds on the minimum distance of linear codes. http://www.codetables.de
22. Grassl, M.: Bounds on the minimum distance of entanglement-assisted qutrit quantum codes. http://www.codetables.de/EAQECC/qutrit.php

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.