Ex.: Find the directional derivative $D_u f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{u}$ is the unit vector given by angle $\theta = \frac{\pi}{6}$. What is $D_u f(1,2)$?

$$\nabla f = \langle 3x^2 - 3y, -3x + 8y \rangle$$
$$\nabla f\Big|_{(1,2)} = \langle -3, 13 \rangle$$

$$\vec{u} = \cos\frac{\pi}{6}\,\vec{i} + \sin\frac{\pi}{6}\,\vec{j} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \rightarrow \text{unit}$$

$$D_u f(1,2) = \nabla f\Big|_{(1,2)} \cdot \vec{u} = -\frac{3\sqrt{3}}{2} + \frac{13}{2} = \frac{13 - 3\sqrt{3}}{2}.$$

Ex.: If $f(x,y,z) = x\sin(yz)$ find the directional derivative of f at $(1,3,0)$ in the direction of $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$.

$$\nabla f = \langle \sin(yz), xz\cos(yz), xy\cos(yz) \rangle$$
$$\nabla f\Big|_{(1,3,0)} = \langle 0, 0, 3 \rangle$$
$$|\vec{v}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$
$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$$
$$D_u f(1,3,0) = \langle 0, 0, 3 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle = -\frac{3}{\sqrt{6}} = \frac{-3\sqrt{6}}{6} = \frac{-1}{2}\sqrt{6}$$

Ex.: Find the directional derivative of $f(x,y) = \sqrt{xy}$ at $P(2,8)$ in the direction of $Q(5,4)$.

$$\vec{PQ} = \langle 3, -4 \rangle \qquad \vec{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

$$\nabla f = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle \Rightarrow \nabla f\Big|_{(2,8)} = \left\langle \frac{4}{4}, \frac{2}{2\cdot 4} \right\rangle = \left\langle 1, \frac{1}{4} \right\rangle$$

$D_u f(2,8) = \langle 1, \frac{1}{4} \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}$

__Ex.__: Find the max. rate of change of $f(x,y) = 4y\sqrt{x}$ at the point $(4,1)$.

$D_u f = \nabla f \cdot u = |\nabla f| \cdot \underbrace{|u|}_{1} \cdot \cos\theta = |\nabla f| \cdot \cos\theta$

$\theta = 0 \Rightarrow \vec{u} = \frac{\nabla f}{|\nabla f|}$ $\quad$ $\nabla f = \langle \frac{2y}{\sqrt{x}}, 4\sqrt{x} \rangle \Rightarrow \langle 1, 8 \rangle$

$|\nabla f| = \sqrt{1+8^2} = \sqrt{65}$ $\quad$ $\vec{u} = \langle \frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}} \rangle$

max. rate of change: $|\nabla f| = \sqrt{65}$ (positive)

directional derivative

__Ex.__: If $f(x,y) = xe^y$, find the rate of change at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$. In what direction does f have the max. rate of change? What is this max. rate of change?

$\overrightarrow{PQ} = \langle -\frac{3}{2}, 2 \rangle \Rightarrow |\overrightarrow{PQ}| = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$

$\vec{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$2 \cdot \overrightarrow{PQ} = \langle -3, 4 \rangle \Rightarrow |2\overrightarrow{PQ}| = 5$

$\vec{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$\nabla f = \langle e^y, xe^y \rangle$

$\nabla f \big|_{(2,0)} = \langle 1, 2 \rangle$

$D_u f(2,0) = \langle 1, 2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = -\frac{3}{5} + \frac{8}{5} = 1$

In the direction of $\nabla f = \langle 1, 2 \rangle$ f has the max. rate of change. $|\nabla f| = \sqrt{1+4} = \sqrt{5}$ is the max. rate of change.

<u>Ex.</u>: Find all points at which the direction of fastest change of the function $f(x,y)=x^2+y^2-2x-4y$ is $\underline{\vec{i}+\vec{j}}$.

$\nabla f=\langle 2x-2,\ 2y-4\rangle \Rightarrow \nabla f=k.\langle 1,1\rangle$ (k: constant)

$\langle 2x-2,\ 2y-4\rangle = k\langle 1,1\rangle \Rightarrow 2x-2=k\ ,\quad 2y-4=k$

$\Rightarrow \cancel{2}x-\cancel{2}=\cancel{2}y-\cancel{4}^{2} \Rightarrow x-1=y-2 \Rightarrow y=x+1$ all points on this line.

<u>Ex.</u>: The second directional derivative of $f(x,y)$ is $D_u^2 f(x,y)=D_u[D_u f(x,y)]$. If $f(x,y)=x^3+5x^2y+y^3$ and $\vec{u}=\langle \frac{3}{5},\frac{4}{5}\rangle$, calculate $D_u^2 f(2,1)$.

$$\nabla f=\langle 3x^2+10xy,\ 5x^2+3y^2\rangle$$

$$D_u f(x,y)=\nabla f.\vec{u}=\frac{9}{5}x^2+6xy+4x^2+\frac{12}{5}y^2=\frac{29}{5}x^2+6xy+\frac{12}{5}y^2$$

$$\nabla(D_u f(x,y))=\langle \frac{58}{5}x+6y,\ 6x+\frac{24}{5}y\rangle \Rightarrow \nabla(D_u f(x,y))\Big|_{(2,1)}=\langle \frac{146}{5},\frac{84}{5}\rangle$$

$$D_u^2 f(2,1)=\langle \frac{146}{5},\frac{84}{5}\rangle.\langle \frac{3}{5},\frac{4}{5}\rangle=\frac{774}{25}.$$

<u>Ex.</u>: Find the equations of the tangent plane and normal line at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$

$$F(x,y,z)=\frac{x^2}{4}+y^2+\frac{z^2}{9}$$

$$\nabla F=\langle \frac{x}{2},\ 2y,\ \frac{2z}{9}\rangle \Rightarrow \nabla F\Big|_{(-2,1,-3)}=\langle -1,2,\frac{-2}{3}\rangle$$

$$-1.(x+2)+2.(y-1)+\left(-\frac{2}{3}\right).(z+3)=0 \Rightarrow 3x-6y+2z+18=0$$

$x=-2+(-1)t\ ,\quad y=1+2t\ ,\quad z=-3+\left(-\frac{2}{3}\right).t$ (normal line)

Ex.: Find the equations of the tangent plane and the normal line to the surface $xy^2z^3=8$ at the point $(2,2,1)$

$\vec{n}=\langle 1,2,6\rangle$

$$F(x,y,z)=xy^2z^3 \Rightarrow \nabla F=\langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2\rangle \Rightarrow \nabla F\Big|_{(2,2,1)}=\langle 4,8,24\rangle$$

$$4\cdot(x-2)+8\cdot(y-2)+24(z-1)=0 \Rightarrow x+2y+6z=12. \text{ (Tangent plane)}$$

$$x=2+t\ ,\quad y=2+2t\ ,\quad z=1+6t \quad \text{(Normal line)}$$

Ex.: Find an equation of the tangent plane to the surface $z=x\sin(x+y)$ at the point $(-1,1,0)$.

$$\boxed{F(x,y,z)=x\sin(x+y)-z}$$

$$z_x=\sin(x+y)+x\cdot\cos(x+y)\ ,\qquad z_y=x\cdot\cos(x+y)$$

$$z_x\Big|_{(-1,1)}=-1\ ,\qquad z_y\Big|_{(-1,1)}=-1$$

$$-1\cdot(x+1)-1(y-1)-(z-0)=0 \Rightarrow x+y+z=0$$

Ex.: Find parametric equations to the tangent line to the curve of intersection of the paraboloid $z=x^2+y^2$ and the ellipsoid $4x^2+y^2+z^2=9$ at the point $(-1,1,2)$.

$$F(x,y,z)=x^2+y^2-z \Rightarrow \nabla F=\langle 2x,2y,-1\rangle \Rightarrow \nabla F\Big|_P=\langle -2,2,-1\rangle$$

$$G(x,y,z)=4x^2+y^2+z^2 \Rightarrow \nabla G=\langle 8x,2y,2z\rangle \Rightarrow \nabla G\Big|_P=\langle -8,2,4\rangle$$

$$=\langle -4,1,2\rangle$$

$$\vec{a}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ -4 & 1 & 2 \end{vmatrix}=\langle 5,8,6\rangle$$

$$\begin{cases} x=-1+5t \\ y=1+8t \\ z=2+6t \end{cases}$$