NUMERICAL SOLUTIONS OF A CLASS OF NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS IN HERMITE SERIES

by

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The purpose of this paper is to present a Hermite polynomial approach for solving a high-order ODE with non-linear terms under mixed conditions. The method we used is a matrix method based on collocation points together with truncated Hermite series and reduces the solution of equation to solution of a matrix equation which corresponds to a system of non-linear algebraic equations with unknown Hermite coefficients. In addition, to illustrate the validity and applicability of the method, some numerical examples together with residual error analysis are performed and the obtained results are compared with the existing result in literature.

Key words: *Hermite polynomials and series, non-linear ODE, matrix and collocation method*

Introduction

Non-linear ODE play an important role in many physical and technical applications and are essential tools for modelling many physical situations [1-9] such as chemical reactions, spring-mass systems, bending of beams and so forth. Most of these type equations have no analytical solution and numerical methods are required to obtain approximate solutions [1-3]. Recently, authors in [10-15] have presented the matrix and collocation methods for solving linear and non-linear differential and integral equations in terms of special polynomials. In this study, we develop the mentioned matrix and collocation for solving the m^{th}-order ODE with non-linear terms in the form:

$$\sum_{k=0}^{m} P_k(x)\, y^{(k)}(x) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x)\, y^{(p)}(x)\, y^{(q)}(x) = g(x) \tag{1}$$

under the mixed conditions:

$$\sum_{k=0}^{m-1}\left[a_{kj} y^{(k)}(a) + b_{kj} y^{(k)}(b) + c_{kj} y^{(k)}(c)\right] = \lambda_j \tag{2}$$

and apply to find the approximate solution in the truncated Hermite series:

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$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n H_n(x), \quad -\infty < a \le x \le b < \infty \tag{3}$$

Here $P_k(x)$, $Q_{pq}(x)$, and $g(x)$ are the functions defined on the interval $-\infty < a \le \le x \le b < \infty$; a_{kj}, b_{kj}, c_{kj}, and λ_j are appropriate constants, a_n $(n = 0,1,\ldots,N \ge m)$ are unknown Hermite coefficients to be determined, $H_n(x)$,$n = 0,1,\ldots,N$, are the Hermite polynomials defined:

$$H_n(x) = \sum_{k=0}^{\left[\left[\frac{n}{2}\right]\right]} \frac{(-1)^k n! 2^{n-2k}}{(n-2k)!k!} x^{n-2k} \tag{4}$$

These polynomials are orthogonal on $(-\infty,\infty)$ with respect to weight function $w(x) = \mathrm{e}^{-x^2}$ and generally defined by Rodrigues:

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} (\mathrm{e}^{-x^2})^{(n)} \tag{5}$$

The first few Hermite polynomials can be given as, explicit expressions from eqs. (4) to (5):

$$H_0(x) = 1, \; H_1(x) = 2x, \; H_2(x) = 4x^2 - 2, \; H_3(x) = 8x^3 - 12x, \; H_4(x) = 16x^4 - 48x^2 + 12$$

Also, the sequence of Hermite polynomials, $H_n(x)$, satisfies the recurrence relation:

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{with} \quad H_0'(x) = 0, \quad H_1'(x) = 2H_0(x) \tag{6}$$

Fundamental matrix relations

Let us consider the m^{th}-order non-linear differential eq. (1) and find the matrix forms of each term in the equation. We firstly convert the solution $y(x)$ by a truncated Hermite series (3) to matrix form:

$$y(x) \cong y_N(x) = H(x)A \tag{7}$$

where

$$H(x) = \begin{bmatrix} H_0(x) & H_1(x) & \cdots & H_n(x) \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T$$

Also, by using the expression (6), for $n = 0,1,\ldots,N$, we find the recurrance relation between the matrix $H(x)$ and its derivative $H^{(k)}(x)$ as:

$$H^{(k)}(x) = H(x)M^k, \quad k = 0,1,\ldots,m \tag{8}$$

where

$$M = \begin{bmatrix} 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad M^0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

By using eqs. (7) and (8), we have the matrix relation:

$$y^{(k)}(x) \cong y_N^{(k)}(x) = H^{(k)}(x)A = H(x)M^k A, \quad k = 0,1,\ldots,m \tag{9}$$

In addition, we can obtain the following matrix forms of the expressions $[y^{(0)}(x)]^2$, $y^{(1)}(x)y^{(0)}(x)$, $[y^{(1)}(x)]^2$, $y^{(2)}(x)y^{(1)}(x)$, $y^{(2)}(x)y^{(0)}(x)$ and $[y^{(2)}(x)]^2$, by means of similar operations as (7)-(9) [12, 16]:

$$\left[y^{(0)}(x)\right]^2 = H(x)\overline{H}(x)\overline{A} \tag{10.1}$$

$$y^{(1)}(x)y^{(0)}(x) = H(x)M\overline{H}(x)\overline{A} \tag{10.2}$$

$$\left[y^{(1)}(x)\right]^2 = H(x)M\overline{H}(x)\overline{M}\overline{A} \tag{10.3}$$

$$y^{(2)}(x)y^{(1)}(x) = H(x)M^2\overline{H}(x)\overline{M}\overline{A} \tag{10.4}$$

$$y^{(2)}(x)y^{(0)}(x) = H(x)M^2\overline{H}(x)\overline{A} \tag{10.5}$$

$$\left[y^{(2)}(x)\right]^2 = H(x)M^2\overline{H}(x)\overline{M^2}\,\overline{A} \tag{10.6}$$

where

$$y^{(0)}(x) = y(x), \quad y^{(1)}(x) = y'(x), \quad y^{(2)}(x) = y''(x)$$

$$\overline{H}(x) = \operatorname{diag}\left[H(x) \quad H(x) \quad \ldots \quad H(x)\right]; \overline{H}(x)_{(N+1)\times(N+1)^2}$$

$$\overline{M} = \operatorname{diag}\left[M \quad M \quad \ldots \quad M\right]; \overline{M}_{(N+1)^2\times(N+1)^2}$$

$$\overline{M^2} = \operatorname{diag}\left[M^2 \quad M^2 \quad \ldots \quad M^2\right]; \overline{M^2}_{(N+1)^2\times(N+1)^2}$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}_{(N+1)\times 1}, \quad \overline{A} = \begin{bmatrix} a_0 A \\ a_1 A \\ \vdots \\ a_N A \end{bmatrix}_{(N+1)^2\times 1}$$

Now we can define the collocation points as:

$$x_i = a + \frac{b-a}{N}i \quad i = 0,1,\ldots,N; \quad a \le x_0 < x_1 < \ldots < x_N = b \tag{11}$$

By substituting the collocation points, eq. (11), into eq. (1), we get the system of matrix equations, for $i = 0,1,\ldots,N$:

$$\sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_i)y^{(p)}(x_i)y^{(q)}(x_i) = g(x_i)$$

or the compact form:

$$\sum_{k=0}^{m} P_k Y^{(k)} + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq} Y^{(p,q)} = G \tag{12}$$

where

$$P_k = \operatorname{diag}\left[P_k(x_0) \quad P_k(x_1) \quad \ldots \quad P_k(x_N)\right]$$

$$Q_{pq} = \operatorname{diag}\left[Q_{pq}(x_0) \quad Q_{pq}(x_1) \quad \ldots \quad Q_{pq}(x_N)\right]$$

$$Y^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix}, \quad Y^{(p,q)} = \begin{bmatrix} y^{(p)}(x_0)y^{(q)}(x_0) \\ y^{(p)}(x_1)y^{(q)}(x_1) \\ \vdots \\ y^{(p)}(x_N)y^{(q)}(x_N) \end{bmatrix}, \quad G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}$$

By putting the collocation points (11) into the matrix relation (9), we obtain the matrix equation, for $i = 0,1,\ldots,N$:

$$y^{(k)}(x_i) = H(x_i)M^k A \Rightarrow Y^{(k)} = HM^k A \tag{13}$$

where

$$H = \begin{bmatrix} H(x_0) \\ H(x_1) \\ \vdots \\ H(x_N) \end{bmatrix} = \begin{bmatrix} H_0(x_0) & H_1(x_0) & \ldots & H_N(x_0) \\ H_0(x_1) & H_1(x_1) & \ldots & H_N(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(x_N) & H_1(x_N) & \ldots & H_N(x_N) \end{bmatrix}$$

On the other hand, we can write the non-linear part of eq. (12):

$$\sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}Y^{(p,q)} = Q_{00}Y^{(0,0)} + Q_{10}Y^{(1,0)} + Q_{11}Y^{(1,1)} + Q_{20}Y^{(2,0)} + Q_{21}Y^{(2,1)} + Q_{22}Y^{(2,2)} \tag{14}$$

Besides, by substituting the collocation points (11) into the relations (10.1)-(10.6), the matrices $Y^{(0,0)}$, $Y^{(1,0)}$, $Y^{(2,0)}$, $Y^{(2,1)}$ and $Y^{(2,2)}$ are obtained:

$$\begin{aligned} Y^{(0,0)} &= H^*_{0,0}\overline{A} \quad & Y^{(1,0)} &= H^*_{1,0}\overline{A} \quad & Y^{(1,1)} &= H^*_{1,1}\overline{A} \\ Y^{(2,0)} &= H^*_{2,0}\overline{A} \quad & Y^{(2,1)} &= H^*_{2,1}\overline{A} \quad & Y^{(2,2)} &= H^*_{2,2}\overline{A} \end{aligned} \tag{15}$$

where

$$H^*_{0,0} = \begin{bmatrix} H(x_0)\overline{H}(x_0) \\ H(x_1)\overline{H}(x_1) \\ \vdots \\ H(x_N)\overline{H}(x_N) \end{bmatrix}, \quad H^*_{1,0} = \begin{bmatrix} H(x_0)M\overline{H}(x_0) \\ H(x_1)M\overline{H}(x_1) \\ \vdots \\ H(x_N)M\overline{H}(x_N) \end{bmatrix}$$

$$H^*_{1,1} = \begin{bmatrix} H(x_0)M\overline{H}(x_0)\overline{M} \\ H(x_1)M\overline{H}(x_1)\overline{M} \\ \vdots \\ H(x_N)M\overline{H}(x_N)\overline{M} \end{bmatrix}, \quad H^*_{2,0} = \begin{bmatrix} H(x_0)M^2\overline{H}(x_0) \\ H(x_1)M^2\overline{H}(x_1) \\ \vdots \\ H(x_N)M^2\overline{H}(x_N) \end{bmatrix}$$

$$H_{2,1}^{*}=\begin{bmatrix} H(x_0)M^2H(x_0)\overline{M} \\ H(x_1)M^2\overline{H}(x_1)\overline{M} \\ \vdots \\ H(x_N)M^2\overline{H}(x_N)\overline{M} \end{bmatrix},\quad H_{2,2}^{*}=\begin{bmatrix} H(x_0)M^2\overline{H}(x_0)\overline{M^2} \\ H(x_1)M^2\overline{H}(x_1)\overline{M^2} \\ \vdots \\ H(x_N)M^2\overline{H}(x_N)\overline{M^2} \end{bmatrix}$$

$$\overline{A}=\begin{bmatrix} a_0A & a_1A & \ldots & a_NA \end{bmatrix}^T$$

Hermite matrix collocation method

We now are ready to construct the fundamental matrix equation corresponding to eq. (1). For this purpose, substituting the matrix relations (13)-(15) into eq. (12), we obtain the fundamental matrix equation:

$$\sum_{k=0}^{m}P_kHM^{(k)}A+\sum_{p=0}^{2}\sum_{q=0}^{p}Q_{pq}H_{(p,q)}^{*}\overline{A}=G$$

or the compact form:

$$WA+V\overline{A}=G \tag{16}$$

where

$$W=\sum_{k=0}^{m}P_kHM^k=\left[w_{ij}\right];\quad i,j=0,1,\ldots,N$$

$$V=\sum_{p=0}^{2}\sum_{q=0}^{p}Q_{pq}H_{p,q}^{*}=\left[v_{mn}\right];\quad m=0,1,\ldots,N;n=0,1,\ldots,(N+1)^2-1$$

Also we can write the matrix eq. (16) in the augmented matrix form:

$$\left[W;V:G\right] \tag{17}$$

or clearly:

$$\left[W;V:G\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N}; & v_{00} & v_{01} & \cdots & v_{0(N+1)^2-1}: & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N}; & v_{10} & v_{11} & \cdots & v_{1(N+1)^2-1}: & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN}; & v_{N0} & v_{N1} & \cdots & v_{N(N+1)^2-1}: & g(x_N) \end{bmatrix}$$

Besides, by using the matrix relation (9), the matrix relation for the mixed conditions (2) is obtained:

$$\sum_{k=0}^{m-1}\left[a_{kj}H(a)+b_{kj}H(b)+c_{kj}H(c)\right]M^kA=\left[\lambda_j\right]$$

or briefly:

$$UA+O^{*}\overline{A}=\lambda \quad\Rightarrow\quad \left[U;O^{*}:\lambda\right] \tag{18}$$

or clearly:

$$\left[U;O^{*}:\lambda\right]=\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} & ; & 0 & 0 & \cdots & 0: & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & 0 & 0 & \cdots & 0: & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N}; & & 0 & 0 & \cdots & 0: & \lambda_{m-1} \end{bmatrix}$$

where

$$U=\left[u_{j0}\quad u_{j1}\quad\cdots\quad u_{jN}\right];\quad j=0,1,\ldots,m-1$$

$$=\sum_{k=0}^{m-1}\left[a_{kj}X(a)+b_{kj}X(b)+c_{kj}X(c)\right]B^{k}$$

$$\lambda=\left[\lambda_0\quad\lambda_1\quad\cdots\quad\lambda_{m-1}\right]^{T}$$

$$O^{*}=\left[0\quad 0\quad\cdots\quad 0\right]\ (\text{zero matrix})$$

Consequenlty, to find Hermite coefficients a_n $(n=0,1,\ldots,N)$ related with the approximate solution (3) of the problem (1) and (2), by replacing the m row matrices (18) by the last m rows (or any m rows) of the augmented matrix (17), we obtain the resulting matrix:

$$\left[\widetilde{W};\tilde{V}:\tilde{G}\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N}; & v_{00} & v_{01} & \cdots & v_{0(N+1)^2-1}; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N}; & v_{10} & v_{11} & \cdots & v_{1(N+1)^2-1}; & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N}; & v_{N-m,0} & v_{N-m,1} & \cdots & v_{N-m,(N+1)^2-1}; & g(x_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & 0 & 0 & \cdots & 0 & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & 0 & 0 & \cdots & 0 & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & 0 & 0 & \cdots & 0 & \lambda_{m-1} \end{bmatrix}$$

From this non-linear system, that is the matrix equation $\tilde{W}A+\tilde{V}\overline{A}=G$, the unknown coefficients a_n $(n=0,1,\ldots,N)$ are determined; therefore the truncated Hermite series solution (3) is obtained:

$$y_N(x)=\sum_{n=0}^{N}a_nH_n(x)$$

Accuracy of solutions and residual error estimation

We can check the accuracy of the obtained solutions as follows [16]. Since the truncated Hermite series (3) is an approximate solution of eq. (1). When the solution $y_N(x)$ and its derivatives are substituted in eq. (1), the resulting equation must be satisfied approximately; that is, for $x=x_l\in[a,b]$, $l=0,1,2,\ldots$

$$R_N(x_l)=\sum_{k=0}^{m}P_k(x_l)y^{(k)}(x_l)+\sum_{p=0}^{2}\sum_{q=0}^{p}Q_{pq}(x_l)y^{(p)}(x_l)y^{(q)}(x_l)-g(x_l)\cong 0$$

or

$$R_N(x_l) \le 10^{-k_l}, \quad (k_l \text{ is any positive numbers})$$

If $\max 10^{-k_l} = 10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $R_N(x_l)$ at each of the points becomes smaller than the prescribed 10^{-k}. Therefore, if $R_N(x_l) \to 0$ when N is sufficiently large enough, then the error decreases.

On the other hand, by means of the residual function $R_N(x)$ and the mean value of the function $|R_N(x)|$ on the interval $[a,b]$, the accuracy of the solution can be controlled and the error can be estimated [17].

Thus, the upper bound of the mean error $\overline{R}_n$:

$$\left|\int_a^b R_N(x)\mathrm{d}x\right| \le \int_a^b \left|R_N(x)\right|\mathrm{d}x$$

and

$$\left|\int_a^b R_N(x)\mathrm{d}x\right| = (b-a)\left|R_N(c)\right|, \quad a \le c \le b$$

$$\left|\int_a^b R_N(x)\mathrm{d}x\right| = (b-a)\left|R_N(c)\right| \le \int_a^b \left|R_N(x)\right|\mathrm{d}x$$

$$\left|R_N(c)\right| \le \frac{\int_a^b \left|R_N(x)\right|\mathrm{d}x}{b-a} = \overline{R}_n$$

Moreover we use different error norms for measuring errors. These are defined:

– $$L_2 = \sqrt{\sum_{i=0}^{n} (e_i)^2}$$

– $$L_\infty = \max(e_i), \ 0 \le i \le n$$

– $$RMS = \sqrt{\frac{\sum_{i=1}^{n} (e_i)^2}{n+1}}$$

where $e_i = |y(x_i) - y_N(x_i)|$ also y and y_N are the exact and approximate solutions of the problem, respectively [18].

Numerical example

The method of this study is useful in finding the solutions of a class of non-linear equations in terms of Hermite polynomials and the accuracy. We illustrate it by the several numerical examples and perform all of them on the computer using a program written separately in MATLAB R2017b.

Example 1. Consider the second order non-linear differential equation:

$$y''(x) - 2y'(x) + y(x) + y^2(x) - y''(x)y(x) = g(x) \tag{19}$$

with the initial and boundary conditions:

$$y(0)=3,\ \ y'(0)=2 \qquad 0\le x\le 1 \tag{20}$$

where $g(x)=2+2e^{x}$.

While the exact solution is $y(x)=1+2e^{x}$, the proposed method is applied and the approximate solutions of eq. (19) under the conditions (20) are obtained as $y_2(x)=3+2x+x^2$, $y_3(x)=3+2x+x^2+0.3944x^3$, $y_4(x)=3+2x+x^2+0.32544x^3+0.10704x^4$, $y_5(x)=3+2x+$ $+x^2+0.336336x^3+0.074608x^4+0.025772x^5$ for $N=2$-5, respectively.

In tabs. 1 and 2, we see that absolute errors and L_2, L_∞ and *RMS* errors are calculated for $N=2,3,4,5$, respectively. In tab. 3, the exact solution of the problem and the the approximate solutions of the problem obtained with Taylor matrix method [19] and the proposed method on $x\in[0,1]$ for $N=5$, are presented.

Table 1. Absolute error of *Example 1* for N = 2-5 and h = 0.1

	Absolute errors			
x_l	e_2	e_3	e_4	e_5
0	0	0	0	0
0.1	3.4184e–04	5.2564e–05	5.6922e–06	2.21837e–06
0.2	2.8000e–03	3.4968e–04	3.0732e–05	1.27917e–05
0.3	9.7000e–03	9.3118e–04	6.3711e–05	3.04095e–05
0.4	2.3600e–02	1.6100e–03	8.1011e–05	4.99869e–05
0.5	4.7440e–02	1.9000e–03	7.2541e–05	6.78585e–05
0.6	8.4200e–02	9.5280e–04	7.0177e–05	8.42649e–05
0.7	1.3751e–01	2.2000e–03	1.7919e–04	1.02848e–04
0.8	2.1110e–01	9.1000e–03	6.1299e–04	1.26842e–04
0.9	3.3092e–01	2.1700e–02	1.7000e–03	1.51611e–04
1	4.3664e–01	4.22000e02	4.1000e–03	1.53143e–04

Table 2. The L_2, L_∞ and errors of *Example 1* for N = 2-5

N	L_2-error	L_∞-error	*RMS*-error
2	5.997233e–01	4.365636e–01	2.998616e–01
3	4.842227e–02	4.216365e–02	2.421113e–02
4	4.483755e–03	4.083656e–03	2.241877e–03
5	2.973264e–04	1.531430e–04	1.486632e–04

Table 3. Comporison of the solutions of *Example 1* for N = 5 and h = 0.1

x_l	Exact solution	Taylor matrix method	Proposed method
0	3	3	3
0.1	3.21034	3.21034	3.21034
0.2	3.44280	3.44280	3.44281
0.3	3.69971	3.69972	3.69974
0.4	3.98364	3.98365	3.98369
0.5	4.29744	4.29744	4.29751
0.6	4.64423	4.64424	4.64432
0.7	5.02750	5.02751	5.02760
0.8	5.45108	5.45107	5.45120
0.9	5.91920	5.91913	5.91935
1	6.43656	6.43630	6.43671

Example 2. Consider the second order non-linear differential equation of the form:

$$\frac{\mathrm{d}^2y}{\mathrm{d}^2x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+y^2(x)=1-\sin(x) \tag{21}$$

with the initial conditions [20]:

$$y(0)=0,\ \ y'(0)=1 \tag{22}$$

Similarly, while the exact solution is $y(x)=\sin x$, the approximate solutions of (21) under the conditions (22) are obtained as $y_2(x)=x$, $y_3(x)=x-0.163980335781936x^3$, $y_4(x)=x-0.1697171792058040x^3+0.009876360178284074x^4$, $y_5(x)=x-0.166728x^3+$ $+0.000304x^4+0.007936x^5$ for $N=2-5$, respectively.

Considering $N=2$-5, the obtained approximate solutions are compared with the exact solution in fig.1 and the absolute errors are demonstrated in tab. 4. In tab. 5, for $N=5$, the solutions, exact and obtained with Taylor matrix method and suggested method, are compared [19].

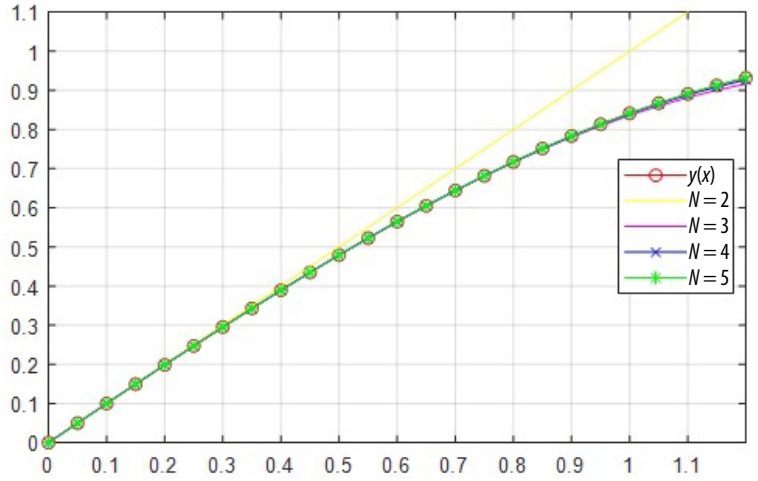


Figure 1 Comporison of the Hermite polynomial solutions and exact solution of *Example 2* for $N=2$-5 (*for color image see journal web site*)

Table 4. Absolute errors of *Example 2* for $N=2$-5 and $h=0.1$

	Absolute errors			
x_n	e_2	e_3	e_4	e_5
0	0	0	0	0
0.1	1.665833e–04	2.603017e–06	2.146190e–06	3.488682e–08
0.2	1.330669e–03	1.882651e–05	1.126605e–05	1.28875e–07
0.3	4.479793e–03	5.232427e–05	2.257198e–05	1.157813e–07
0.4	1.058165e–02	8.691620e–05	2.740695e–05	1.127313e–07
0.5	2.057446e–02	7.691942e–05	2.291349 e–05	4.613957e–07
0.6	3.535752e–02	6.222592e–05	2.140782 e–05	7.803649e–07
0.7	5.578231e–02	4.629424e–04	5.936562 e–05	1.402682e–06
0.8	8.264390e–02	1.314022e–03	2.059295 e–04	4.15998e–06
0.9	1.166730e–01	2.868574e–03	5.708533 e–04	1.396141e–05
1	1.585290e–01	5.451320e–03	1.311803 e–03	4.101519e–05

Table 5. Comporison of the solutions of *Example 2* for $N = 5$ and $h = 0.1$

x_l	Exact solution	Taylor matrix method	Proposed method
0	0	0	0
0.1	0.0998334	0.0998333	0.0998333
0.2	0.1986693	0.1986691	0.1986692
0.3	0.2955202	0.2955199	0.2955201
0.4	0.3894183	0.3894181	0.3894184
0.5	0.4794255	0.4794253	0.4794260
0.6	0.5646424	0.5646418	0.5646432
0.7	0.6442176	0.6442164	0.6442190
0.8	0.7173560	0.7173556	0.7173602
0.9	0.7833269	0.7833332	0.7833408
1	0.8414709	0.8415000	0.8415120

The upper bound of the mean error , $\overline{R}_n$, of *Example 2* can be calculated as $\overline{R}_2 = 7.93031e-01$, $\overline{R}_3 = 4.124611e-02, \overline{R}_4 = 1.305709e-02, \overline{R}_5 = 2.25636e-03$ as in the method given and $\overline{R}_n$ of *Example 2* is illustrated in fig. 2 for N =2-5.

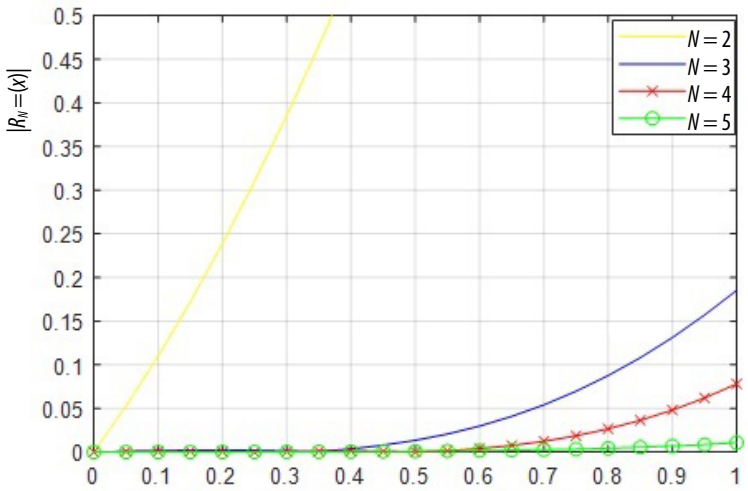


Figure 2. The residual error functions of *Example 2* for N =2-5
(*for color image see journal web site*)

Example 3. Consider the following differential equation:

$$y'(x) - 2y(x) + y^2(x) = g(x)$$
$$g(x) = \begin{cases} (x-1)^2, & 0 \le x \le 1/2 \\ x^2 - 2, & 1/2 < x \le 1 \end{cases} \tag{23}$$

It is solved with the suggested method for the boundary condition:

$$y(0) = 0 \tag{24}$$

For $0 \le x \le 1/2$, $P_0(x) = -2$, $P_1(x) = 1$, $Q_{00}(x) = 1$.

For $g_1(x)=(x-1)^2$, $N=2$ and the ordering points are $x_0=0, x_1=1/4, x_2=1/2$ for the interval $0\le x\le 1/2$.

The solution is investigated in the following form:

$$y(x)\cong\sum_{n=0}^{1}a_n x^n$$

Using our proposed method, the fundamental matrix equation is obtained:

$$P_1HMA+P_0HA+Q_{00}H_{00}^{*}\overline{A}=G$$

Here,

$$P_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad H=\begin{bmatrix}H(0)\\H(1/4)\\H(1/2)\end{bmatrix}=\begin{bmatrix}1&0&-2\\1&\frac{1}{2}&-\frac{7}{4}\\1&1&-1\end{bmatrix}$$

$$Q_{00}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad M=\begin{bmatrix}0&2&0\\0&0&4\\0&0&0\end{bmatrix}\quad P_0=\begin{bmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{bmatrix}\quad A=\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$$

$$H_{00}^{*}=\begin{bmatrix}H(0)\overline{H}(0)\\H(1/4)\overline{H}(1/4)\\H(1/2)\overline{H}(1/2)\end{bmatrix}=\begin{bmatrix}1&0&-2&0&0&0&-2&0&4\\1&1/2&-7/4&1/2&1/4&-7/8&-7/4&-7/8&49/16\\1&1&-1&1&1&-1&-1&-1&1\end{bmatrix}$$

$$\overline{H}(x)=\operatorname{diag}\begin{bmatrix}H(x)&H(x)&H(x)\end{bmatrix}$$

$$A=\begin{bmatrix}a_0&a_1&a_2\end{bmatrix}^T,\quad \overline{A}=\begin{bmatrix}a_0A&a_1A&a_2A\end{bmatrix}^T$$

$$G_1(x)=\begin{bmatrix}g_1(0)&g_1(1/4)&g_1(1/2)\end{bmatrix}^T=\begin{bmatrix}1&9/16&1/4\end{bmatrix}^T$$

The following fundamental matrix equation is obtained:

$$\begin{bmatrix}-2&2&4\\-2&1&\frac{11}{2}\\2&0&6\end{bmatrix}A+\begin{bmatrix}1&0&-2&0&0&0&-2&0&4\\1&\frac{1}{2}&-\frac{7}{4}&\frac{1}{2}&\frac{1}{4}&-\frac{7}{16}&-\frac{7}{8}&-\frac{7}{16}&\frac{49}{64}\\1&1&-1&1&1&-1&-1&-1&1\end{bmatrix}\overline{A}=\begin{bmatrix}1\\\frac{9}{16}\\\frac{1}{4}\end{bmatrix}$$

The $\begin{bmatrix}1&0&-2\end{bmatrix}A+\begin{bmatrix}0&0&0&0&0&0&0&0&0\end{bmatrix}\overline{A}=0$ is obtained for the boundary condition $y(0)=0$ and using the proposed method, the desired matrix equation:

$$\begin{bmatrix}-2&2&4\\1&0&-2\\-2&0&6\end{bmatrix}A+\begin{bmatrix}1&0&-2&0&0&0&-2&0&4\\0&0&0&0&0&0&0&0&0\\1&1&-1&1&1&-1&-1&-1&1\end{bmatrix}\overline{A}=\begin{bmatrix}1\\0\\1/4\end{bmatrix}$$

is constructed.

Thus, Hermite's coefficients are found as $a_0 = 0, a_1 = 1/2, a_2 = 0$. For $0 \le x \le 1/2$, the solution:

$$y(x) = 1/2x, \ \ 0 \le x \le 1/2$$

is obtained.

For $g_1(x) = x^2 - 2, \ N = 2$ and the ordering points are $x_0 = 1/2, \ x_1 = 3/4, \ x_2 = 1$ for the interval $1/2 < x \le 1$.

The $y(1/2) = 1/4$ is obtained for $x_0 = 1/2$ by considering $y(x) = 1/2x$. Thus, the following matrix equation is written:

$$\begin{bmatrix} -2 & 0 & 6 \\ 1 & 1 & -1 \\ -2 & -2 & 4 \end{bmatrix} A + \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 4 & 4 & 2 & 4 & 4 \end{bmatrix} \overline{A} = \begin{bmatrix} -7/4 \\ 1/4 \\ -1 \end{bmatrix}$$

Thus, since there are no real Hermite's coefficients, there is no approximate solution for the interval $1/2 < x \le 1$.

The solution of (23) under the conditions (24):

$$y(x) = \begin{cases} \dfrac{1}{2x}, & 0 \le x \le 1/2 \\ \varnothing, & 1/2 < x \le 1 \end{cases}$$

overlaps with the exact solution of the problem.

Conclusion

Non-linear differential equations used in engineering, physics, mathematics or in many modelling problems are usually difficult to solve as analytically. In this study, to solve a class of non-linear differential equations with boundary conditions, we introduce a matrix method depending on Hermite polynomials and collocation points. Also the residual error analysis has been also developed for the accuracy of solutions. The present method and the error analysis procedures are applied to some examples which have been solved by Taylor matrix method in the literature. The results related with examples have been shown in tabs. 1-5 and figs. 1 and 2. As it is seen from the numerical examples, the method provides a better approximation than the other methods such as Taylor matrix method. A significant advantage of the proposed method, the Hermite coefficients of the solution can be found obviously by developing computer programs. The method also can be developed and applied to solve other high order non-linear differential equations with new strategies.

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