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Development and Verification of hp-FEM for Elastostatic Plane Problems

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Abstract

The complexity of engineering problems hinders analytical approaches and leads to the use of numerical methods, one of which is Finite Element Method (FEM). FEM has come to an undeniable point as an attractive numerical analysis method in the scope of Engineering Design. A lot of studies that FEM has been successfully employed have been conducted. In general low order finite elements are used for the reliable simulation of structural mechanics. Unfortunately, the pure displacement based elements tend to locking effects which cause significant calculation errors. In order to avoid these problems of low order elements, several element techniques, e.g. the assumed natural strain approach and the enhanced assumed strain concept, have been developed. In the present study higher order finite elements as an alternative approach are investigated. The p-finite element method avoids locking naturally within a pure displacement based element formulation. Higher order finite elements based on Lagrange (p=1,…,5), Serendipity (p=2,3,4) and Hierarchic Legendre (p=2,3,4) shape functions are compared to the classical h-method using low order elements and fine meshes. The analyses are performed by a research oriented finite element program developed in MATLAB. Several examples are showing the application to linear structural mechanics, the correctness of the program and the properties of the investigated p-finite element methods. It is observed that the program gets satisfactory results in both displacement and stress field with ANSYS and analytical way. P-method provides more qualified results than h-method in both coarse and fine mesh.

Keywords: Lagrange element, Serendipity element, Hierarchic Legendre element, High order element, Elastostatic plane problem, hp-FEM.

Introduction

Finite Element Method (FEM) is a framework based on discretization of a problem domain into subdomain, where the mapping of coordinates and representation of unknown field are performed on element level. These operations are intimately related

to the effectiveness or efficiency of a FEM since the order of correctness of the obtained results from FEM are desired to approach the intended analytic results. Depending on the problem, the unknowns, in general, become pure displacements and low order elements are preferred to assemble the form of FEM over pure displacements. Unfortunately, the pure displacement based elements tend to locking effects which cause significant calculation errors. In order to avoid these problems of low order elements, several element techniques, e.g. the assumed natural strain approach (Belytschko et al., 2000) and the enhanced assumed strain concept (Simo and Rifai, 1990), have been developed. As an alternative approach, high order elements can be considered because it is naturally qualified within a pure displacement based element formulation in order to circumvante these effects. In this sense, the present study focuses on the higher order elements. The concept of the present study makes comparison of the variations of p-method, such as Lagrange (p=1,…,5), Serendipity (p=2,3,4) and Hierarchic Legendre (p=2,3,4) shape functions, with the classical h-method using low order elements and fine meshes. The analyses are performed by a research oriented finite element program developed in MATLAB. With two examples the application to linear structural mechanics, the correctness of the program and the properties of the investigated p-finite element methods are observed.

Finite Element Procedure

The procedure of the followed FEM is drawn by principle of virtual work (PVW). The generation of PVW consists of two stages. In the first stage, differential equations of balance momentum and of static boundary condition, called as strong form, are multiplied by a test function and the multiplications are integrated over the domain. So the integral expressions are obtained, which are called as weak form. As test function, the virtual displacement δu is chosen. In the second stage, the weak formulations are put together. The weak formulations provide convenience so that the strong form necessiates that differentail equations must be met at every material point, but the weak form requires that they must be met only in an average sense (Cook et al., 1989). Thus, the integral expression allows exact solution of problem to be replaced by an approximated solution, which satisfies the integral but not the local form of the corresponding differential equation (Kuhl and Meschke, 2004). This procedure is elaborated in the next four sub-sections.

Principle of Virtual Work

The PVW is derived in a way of sum of the weak formulation of the balance of momentum and of static boundary condition in tensor notation, and is given in common sense as

$$\int_{\Omega} \delta u \cdot \ddot{u} \rho \, dV + \int_{\Omega} \delta \varepsilon : \sigma \, dV = \int_{\Omega} \delta u \cdot b \rho \, dV + \int_{\Gamma_\sigma} \delta u \cdot t^* \, dA \qquad (1)$$

where u is displacement field of system, δu variation of displacement field, ρ density, δε variation of strain tensor, σ Caughy stress tensor, $\ddot{u}$ acceleration of system, b acceleration belonging to body force in system, t^* vector on the surface of system, Ω material body or domain , Γ_σ static boundary of domain. The right-hand side in Eq. (1) is described as external virtual work δW_{ext}, while the first integral on the left-hand side as virtual work of internal forces δW_{dyn} and the second integral as internal virtual work δW_{int}. Moreover, the strain tensor ε is concieved as linear part of Green Langrange strain tensor and the stress σ as Caughy stress tensor. From static view, virtual work of

internal force δW_{dyn} vanishes, and when only surface loads are considered the second integral term in external virtual work δW_{ext} and so does the body force. So PVW form is reduced to

$$\int_{\Omega} \delta\varepsilon : \sigma dV = \int_{\Gamma_\sigma} \delta u \cdot t^* dA \tag{2}$$

As the tensor form of stress and strain in Eq. (2) are reduced to vector form for convenience so that the matrix notation could be used,

$$\delta W_{int} = \int_{\Omega} \delta\varepsilon \cdot \sigma dV \tag{3}$$

is obtained, where stress and strain are no longer tensor but vector.

Definition of Elastostatic Plane Problem

The formulation of plane problem is generated by degenerative approach. The approach subjects to conversion of the definition of problem throughout its whole volume into a middle surface and thickness h. Thus the structural continuum is represented by the middle surface with the thickness so that the problem can be modeled by plane finite elements. Geometrical description of two-dimensional element is given by position vector having two components each of which lies along two perpendicular axes, respectively.

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^T \tag{4}$$

For plane strain and stress state, it is enough to define independent three components for both stress and starin vector,

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{bmatrix}^T \tag{5}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & 2\varepsilon_{12} \end{bmatrix}^T \tag{6}$$

and constitutive equation of plane stress that refers to stress-starin relationship is formed by means of the constitutive matrix C defined by the elasticity modulus E and poisson ratio υ.

$$\sigma = C\,\varepsilon,\ C = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ & 1 & 0 \\ \text{sym} & & (1-\upsilon)/2 \end{bmatrix} \tag{7}$$

The deformation of two-dimensional continuum is exppressed by the displacement vector having same manner with position vector.

$$U(X_1, X_2) = U(X) = \begin{bmatrix} U_1(X_1, X_2) & U_2(X_1, X_2) \end{bmatrix}^T \tag{8}$$

The relation between strain and displacement or kinematic equation is defined by means of adoptation of linear part of Green Langrange strain tensor as matrix form in the following

$$\varepsilon = \nabla^{sym} u \ , \ \nabla^{sym} u = \frac{1}{2}(\nabla u + \nabla^{T} u) , \tag{9}$$

$$\varepsilon = D_{\varepsilon} u \ , \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial X_1} & 0 \\ 0 & \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{10}$$

In Eq.(10) the matrix D is called as differential operator. Substitution of Eq.(7) and (10) into Eq. (3) yields δW_{int} in alternative form

$$\delta W_{int} = \int_A \delta u \cdot D_{\varepsilon}{}^{T} C D_{\varepsilon} u \, dA . \tag{11}$$

Finite Element Discretization

The discretization of structure domain to finite element is performed by modeling it as a set of finite elements, which is pinned together at node points, and so the structure domain is able to be comprised by an assemblage of small parts (elements). The concept of that not only obtains application of finite element methods, and also alleviates inhibition of the geometry of complex problem by means of elements possesing simple geometry. During finite element procedure, the fundamental approach is that the continuum is broken down and node points originate as presences to approximate continuum. The node points is able to accomplish this task with accompaniment of shape functions first at element level and then at whole problem level in the sense that the shape functions are of tools to create approximation of the continuation. After the discretization is applied to displacement, position, strain, and stress field in terms of isoparametric approximation, the vector forms of these quantities compose of these discretized nodal values. Substition of discrete forms along with inclusion of natural coordinates ξ into the concept of numerical integration, morover, gives the next form of δW_{int} as

$$\delta W_{int}{}^{e} = \int_{-1}^{1}\int_{-1}^{1} \delta u^{e} \cdot B^{T}(\xi) C B(\xi) u^{e} \left| J(\xi) \right| h d\xi_1 d\xi_2 \tag{12}$$

where superscript e points to nodal values of corresponding quantities, and the matrix B is partitional operator and it is obtained by inserting of shape functions into differential operator D and applying of Jacobian transformation. Since both element displacement vector u^e and its variation δu^e are free from the natural coordinates ξ, both of these vectors can be removed from the integral and so the integral expression refers to stiffness matrix k^e.

$$\delta W_{int}{}^{e} = \delta u^{e} \cdot \int_{-1}^{1}\int_{-1}^{1} B^{T}(\xi) C B(\xi) \left| J(\xi) \right| h d\xi_1 d\xi_2 u^{e} = \delta u^{e} \cdot k^{e} u^{e} \tag{13}$$

Shape Functions For Plane (Quadrilateral) Elements

Shape functions, in common sense, are polynomials possessing interpolation property that enable to describe approximate continuum in element level with nodal

contributions. The development of the polynomial bases on the well-known Pascal Triangle. With help of the terms in Pascal Triangle and these coefficients, the polynomials like Lagrange polynomials can be constituted. Depending on position of nodes in element, each node on the element refers to a equation, where unknowns are coeffcients of the terms, and all equations assemble so-called Vandermonde matrix. After solution of this matrix, the coeffcient are found and so the polynomial is obtained. In two-dimesional framework, shape functions of plane (quadrilateral) elements are introduced by means of use of the same polynomials on representation of two natural coordinates ξ_1 and ξ_2, called as bipolynomial. Within conventional way, the bipolynomials are generated by multiplication of two one-dimensional polynomials. In particular, Lagrange interpolation polynomial is used to comprise such kind of bipolynomial and it can be formed by

$$N^i(\xi) = \prod_{\substack{k=1 \\ k\neq i}}^{p+1} \frac{\xi^k - \xi}{\xi^k - \xi^i} \quad , \quad N^i(\xi_1, \xi_2) = \left(\prod_{\substack{k=1 \\ k\neq i}}^{p+1} \frac{\xi_1^k - \xi_1}{\xi_1^k - \xi_1^i} \right) \left(\prod_{\substack{m=1 \\ m\neq i}}^{p+1} \frac{\xi_2^m - \xi_2}{\xi_2^m - \xi_2^i} \right) \tag{14}$$

where p is degree of the bipolynomial, k and m are position numbers of node in the direction of natural coordinates ξ_1 and ξ_2, respectively. $\xi^k{}_1$ and $\xi^m{}_2$ characterize the position of node in the respective directions and N^i is the shape function associated with node i. In Eq. (14), all term in Pascal Triangle, whose potencies are greater than or equal to p, are included and so the shape functions having this form are called as complete Lagrange shape functions or Lagrange shape functions. In view of computational effort and the number of nodes, these shape functions seem expensive and it leads to use reduced Lagrange shape function, i.e. expensive terms vanish. This type of the shape function is known as Serendipity (Kuhl and Meschke, 2004; Hughes, 2000). Figure 1 illustrutes Lagrange and Serendipity elements in Pascal Triangle

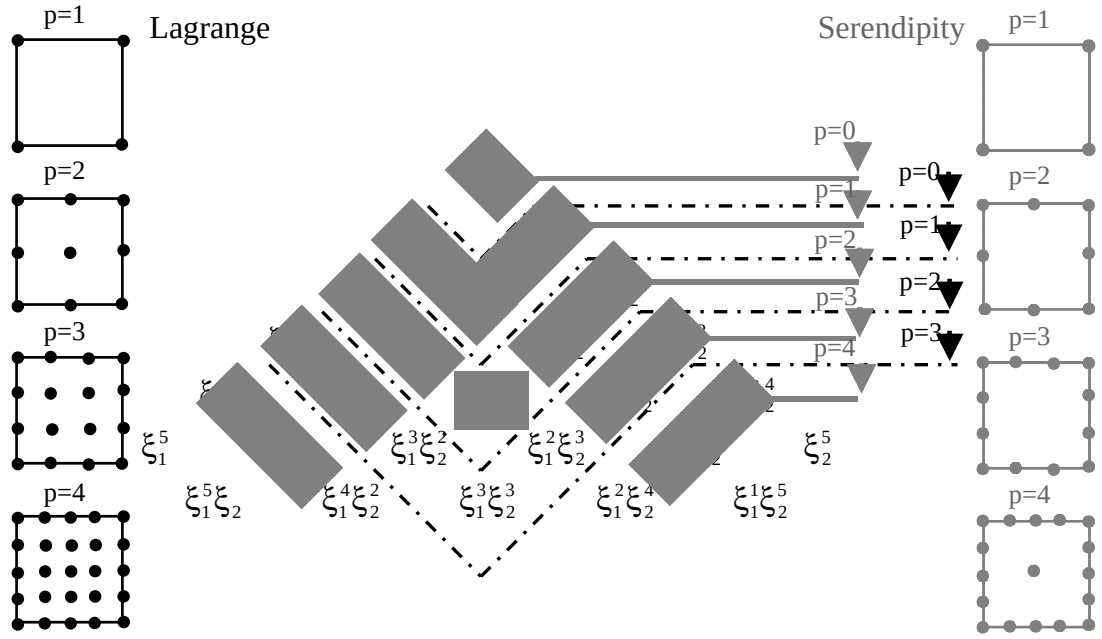
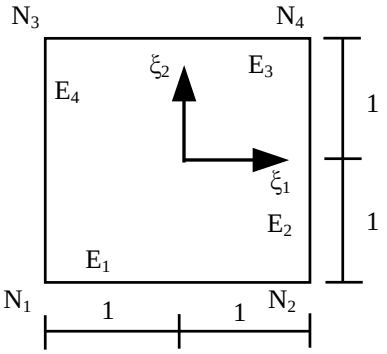


Figure 1. Two-dimensional Lagrange and Serendipity terms of quadrilateral elements in the Pascal Triangle.

In the Serendipity element, inner element nodes are removed until p is equal to 4 and also in order to gurantee at least complete polynomials, serendipity shape functions have to be equipped with these nodes as p is greater than or equal to 4. Serendipity

shape function can be produced by solving Vandermonde matrix of cofficients of coresponding terms in Pascal Triangle. Having ability to construct shape function, another polynomial is Legendre polynomial. The polynomial provides improvement of Hierarchic shape functions (Szabó, and Babuška, 1991; Zienkiewicz et al., 1989), regardless of position of nodes except for corner nodes. Accordingly, computational effort is alleviated thanks to requirement of less node, that is to say, only four corner nodes are enough to define an element with any degree p. Hierarchic shape functions are organized into three categories: nodal shape functions N^{N_i}, side modes N^{E_s} (s is edge number), and internal modes N^{int}. With help of Figure 2, the shape function can be written as



$$N^{N_i} = \frac{1}{4}\left(1+\xi_1^i\xi_1\right)\left(1+\xi_2^i\xi_2\right) \quad i=1, ..., 4$$

$$N^{E_1} = \frac{1}{2}\left(1 \quad \xi_2\right)\phi_i\left(\xi_1\right) \qquad i=2, ..., p \quad (p\geq 2) \qquad (15)$$

$$N^{int} = \phi_i\left(\xi_1\right)\phi_j\left(\xi_2\right)$$

$$i=2, ..., p_1, j=2, ..., p_2, \; p_{1+}\, p_2 = p \;\; (p\geq 4)$$

Figure 2. The standart quadrilateral element.

where p_1 and p_2 are polynomial degrees in respective direction. With the degrees $p \leq 4$, the illustration of the nodal, side, and internal modes of the shape function is also given in Figure 3.

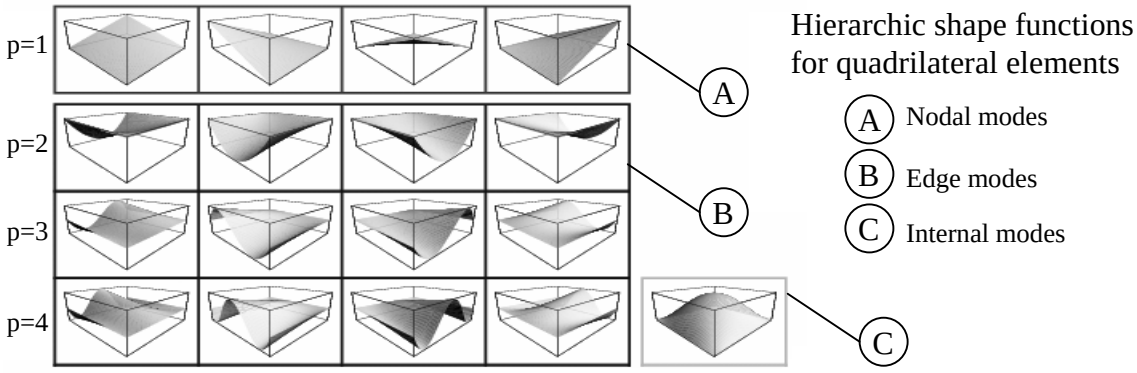


Figure 3. Hierarchic shape functions for quadrilateral element (Szabó, and Babuška, 1991).

Shape functions are used for mapping of coordinates and approximation of displacement field. In this study the mapping and approximation of displacement are represented by the same shape functions, and so this mapping is known as isoparametric mapping. In the developed program the three type elements, such as Lagrange, Serendipity, and Hierarchic Legendre elements, are presented to user as alternative elements. Through the definition of plane problem, user requires only quadrilateral elements consisting of four-nodes. Depending on the types of defined elements, if necessary, elements are furnished with secondary nodes and therefore, there is no need to user intervention as well.

Numerical Tests

A simple beam and notched plane problem are considered as numerical tests. In this study the high order finite elements varied by the considered shape function types are compared to h-method using low order elements and fine mesh as well as the accuracy of the results provided by the current program in MATLAB is verified by analytical and ANSYS solutions. The first test is conceived by regular mesh while the second by non-regular mesh. Both tests, moreover, are modeled by two meshes, one of which is relatively fine to the other.

Simple beam

A simple beam is illustareted in Figure 4.

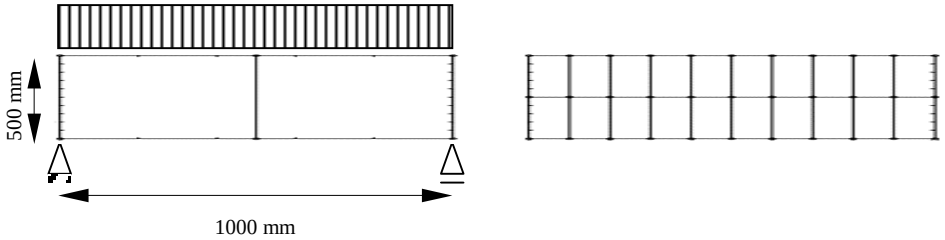


Figure 4. A simple beam (a) with 2 elements (b) with 20 elements under distributed uniform load.

The beam is loaded by uniform distributed load in gravity direction through the all upper bound whose value is 0.1 N/mm. The thickness of the beam is 100 mm, its poisson ratio is 0.2 and its elasticity modulus is 1000 N/mm^2. The problem is modelled by 2 and 20 elements and the type of element is changed to 9-nodes (p=2), 16-nodes (p=3), 25-nodes (p=4) and 36-nodes (p=5) Langrange element; 8-nodes (p=2), 12-nodes (p=3), and 17-nodes (p=4) Serendipity and 4-nodes (p=2, 3, 4) Hierarchic elements, respectively. The obtained results of displacements and Von Mises stresses on the midspan of the bottom bound for current program, ANSYS and analytical way are given in Figure 5 and 6 with respect to the polynomial degree and the type of shape function, and the displacements are also given in Table 1.

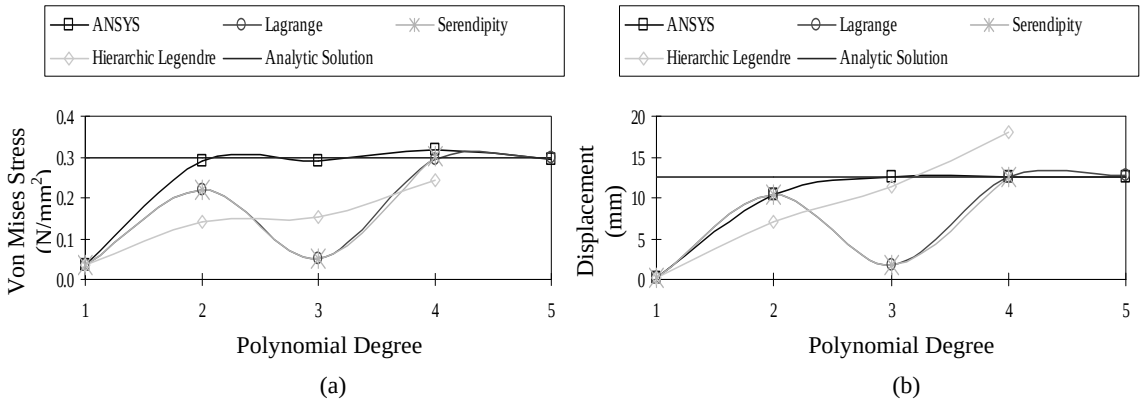
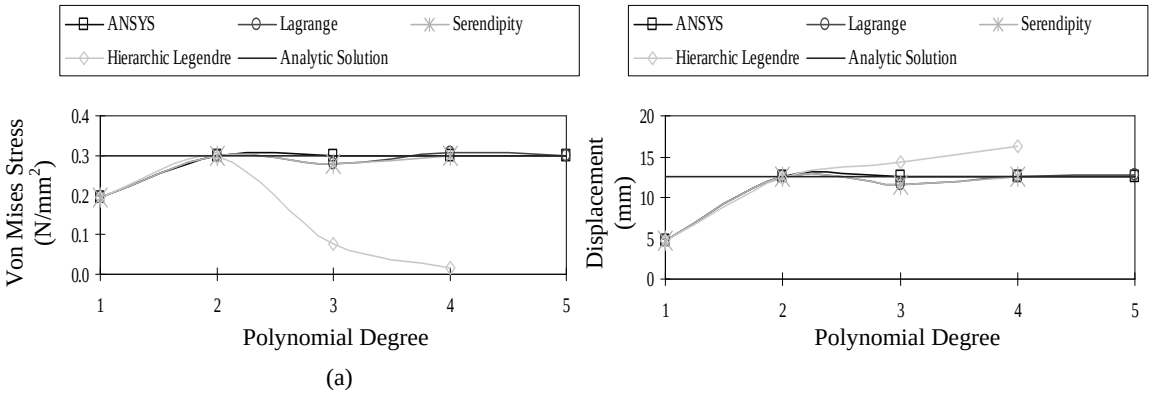


Figure 5. (a) Von Mises Stress (b) Displacement of simple beam with respect to polynomial degree and type of shape function with 2 elements.



(a)

Figure 6. (a) Von Mises Stress (b) Displacement of simple beam with respect to polynomial degree and type of shape function with 20 elements.

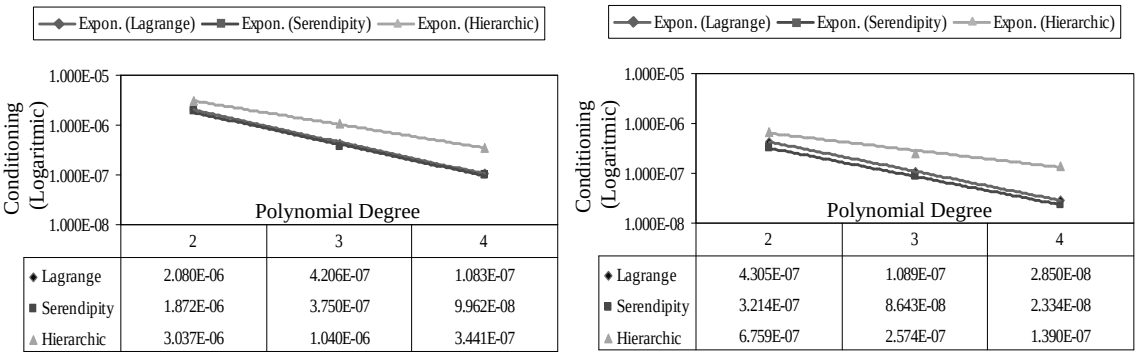
Polynomial Degrees	1	2	3	4	5
			With 2 elements (mm)		
ANSYS	-0.23633	-10.32800	-12.59600	-12.56300	-12.58000
Current program (Lagrange)	-0.23633	-7.06075	-1.76106	-12.55385	-12.67295
Current program (Serendipity)		-10.32778	-1.76053	-12.56648	
Current program (Hieararchic)		-10.32778	-11.42941	-18.05686	
			With 20 elements (mm)		
ANSYS	-0.465430	-12.54200	-12.57700	-12.57900	-12.58000
Current program (Lagrange)	-0.465428	-12.54460	-11.60224	-12.53630	-12.71888
Current program (Serendipity)		-12.54202	-11.60061	-12.59349	
Current program (Hieararchic)		-12.33247	-14.25896	-16.19438	

Table 1. Comparison of maximum displacements on the simple beam with 2 and 20 elements, respectively.

The results of maximum displacement in the bottom middle point of the beam for the current program and ANSYS can be seen from Table 1 so that those are quite close to each other. As can be seen in Figure 5 (b) the displacements obtained for all shape function types converge to those obtained from analytical way. But the convergence is not as smooth as that of ANSYS. In degree p=3, the displacements for Lagrange and Serendipity elements go far from analytical displacement but this divergence disappears as long as mesh becomes fine, which can be seen in Figure 6 (b). The displacement coming from Hieararchic Legendre element converges relatively slowly in coarse mesh compared to those from Lagrange and Serendipity elements but is smoother than those. In fine mesh it also convergences like those from Lagrange and Serendipity elements. The von Mises stresses obtained for all shape function types seem analogous behavior as in displacements in coarse mesh, which is seen in Figure 5. The stress results are also satisfactory in relatively fine mesh except for that of Hieararchic Legendre with degree p=3 and 4, which is seen in Figure 6 (a).

In Figure 7, conditioning κ, a criterion of performance of shape functions, is illustrated. The criterion depends on the ratio of the smallest eigenvalue $\lambda_{\min}$ of global stiffness to the largest eigenvalue $\lambda_{\max}$ of that and is desired to come at 1 in the best case. The condition of system stiffness matrix gives a measure of the approximation regularity of the system which allows evaluation of shape functions.

$$\kappa = \frac{\lambda_{\min}}{\lambda_{\max}} \leq 1 \tag{16}$$



(a) (b)

Figure 7. The graphic of conditioning with respect to the number of polynomial degree in the model (a) with 2 elements (b) with 20 elements.

As can be seen Figure 7 (a), maximum conditioning belongs to Hierarchic Legendre element, and Lagrange and Serendipity elements have more or less identical conditioning. In Figure 7(b) the order of conditioning is the same in fine mesh except Lagrange element outperforms Serendipity element slightly. Furthermore, note that conditioning decreases as long as polynomial degree increases.

Notched-Plane

A notched-plane seen in Figure 8 is also solved to indicate the effects of shape function types in non-regular mesh system. Figure 8 also depicts Lagrange and Serendipity elements with 31 elements (coarse mesh) and 93 elements (relatively fine mesh) with respect to several polynomial degree and element type. The plane has a notch in centre, whose radius is 20 mm, and thickness of the plane is 20 mm. The right hand side of this is loaded by the distributed load 20 N/mm. Elasticity Modulus is 200000 N/mm^2 and Poisson ratio is 0.3. In the follow part, the results provided by ANSYS and the program are given in Figure 9 and 10 with respect to respective polynomial degrees, as well.

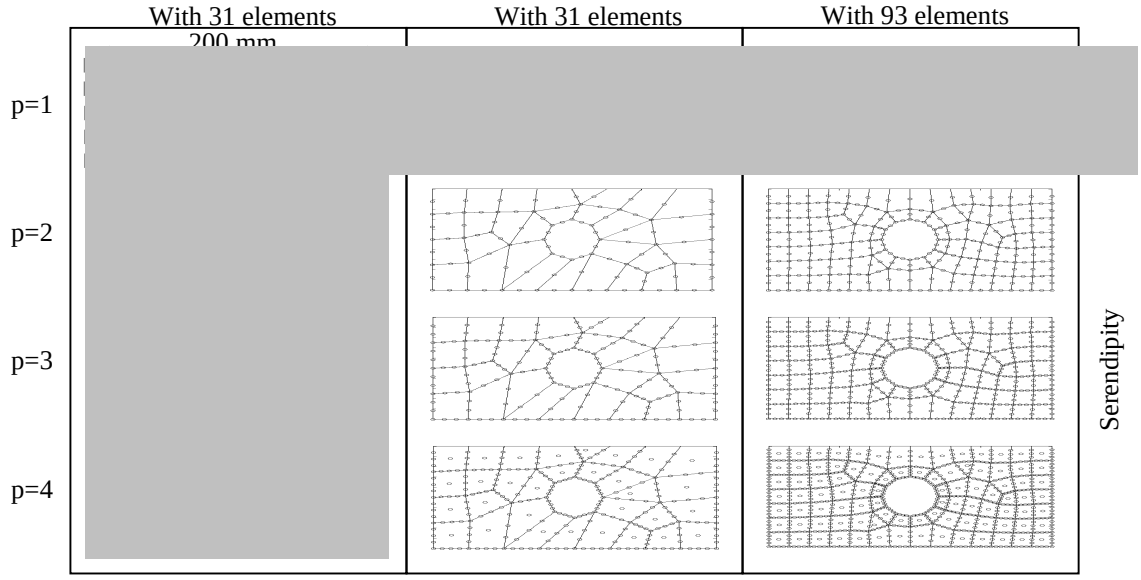


Figure 8. A notched plane (a) with 31 elements (b) with 93 elements under distributed uniform tension load with respect to polynomial degree “p”, and Lagrange and Serendipity elements.

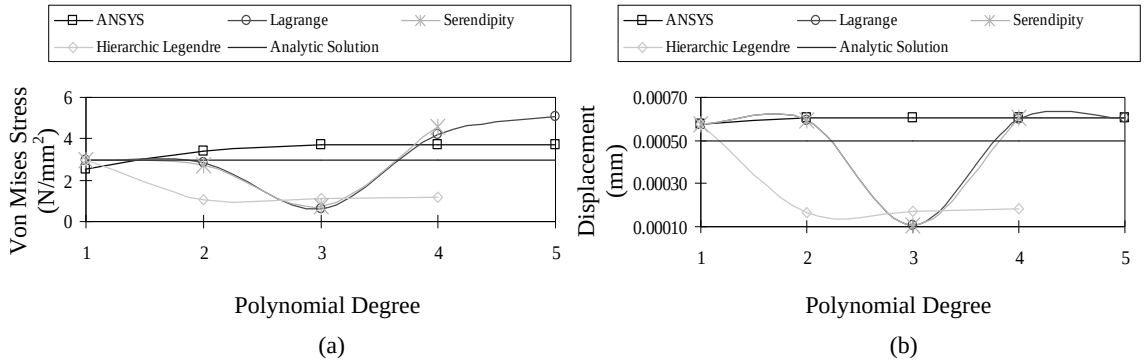


Figure 9. (a) Von Mises Stress (b) Displacement of simple beam with respect to polynomial degree and type of shape function with 31 elements.

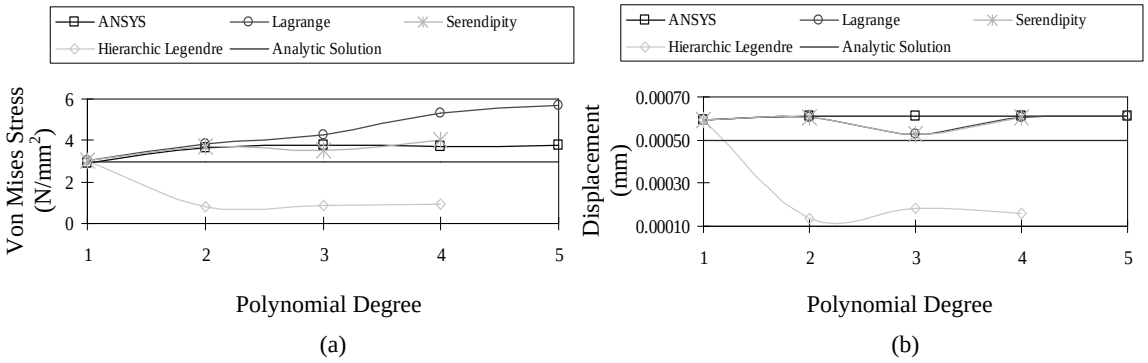
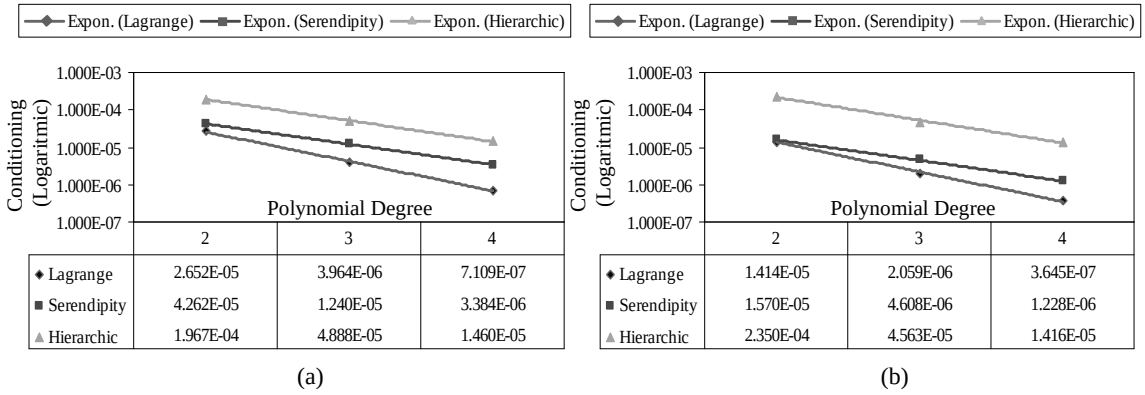


Figure 10. (a) Von Mises Stress (b) Displacement of simple beam with respect to polynomial degree and type of shape function with 93 elements.

In Figure 9 the displacement and stress for Lagrange and Serendipity elements follow similar path with those coming from ANSYS but instantly go down for p=3. As can be seen in Figure 10, the fall of displacement and stress disappear when mesh is fine as in the simple beam. Hierarchic shape function gives unsatisfactory results in both the stress and displacement, which has been expected due to the fact that Hierarchic shape function is inconvenient to map curve boundary. Compared to Serendipity, element Lagrange element overestimates the stress especially for p=3 and 4 in Figure 10 (a). The conditioning for the shape functions is yielded in Figure 11.



	2	3	4
Lagrange	2.652E-05	3.964E-06	7.109E-07
Serendipity	4.262E-05	1.240E-05	3.384E-06
Hierarchic	1.967E-04	4.888E-05	1.460E-05

	2	3	4
Lagrange	1.414E-05	2.059E-06	3.645E-07
Serendipity	1.570E-05	4.608E-06	1.228E-06
Hierarchic	2.350E-04	4.563E-05	1.416E-05

Figure 11. The graphic of conditioning with respect to the number of polynomial degree in the model (a) with 31 elements (b) with 93 elements.

As can be seen Figure 11 (a), Hierarchic Legendre element covers maximum conditioning and Serendipity element demonstrates more efficiency than Lagrange element, regarding of conditioning. The same interference can be made in Figure 11 (b) but Serendipity and Lagrange elements have intimate conditioning for p=2.

Conclusions

In present study high order elements are produced by depending on three shape function types with different polynomial degrees, e.g. Lagrange (p=1,.., 5), Serendipity (p=2, 3, and 4), and Hierarchic Legendre (p=2, 3, and 4) shape functions. With these p version finite elements and h-version finite element, two elastostatic plane problems modeled by coarse and fine mesh are analyzed. One of the problems has regular mesh while the other has non-regular mesh. The obtained displacement and von Mises stress results as well as ANSYS and analytical results are depicted with respect to polynomial degree. Furthermore, conditionings corresponding to the high order elements are evaluated on graphs. In consequence of the study the comparison of p and h methods and also verification of developed program are performed. The interferences are summarized as follows

It is observed that the program gets satisfactory results in both displacement and stress field with ANSYS and analytical way. The increase in the order of element in p-method triggers improvement of the results. P-method provides more qualified results than h-method in both coarse and fine mesh. Moreover, It is noted that Serendipity and Lagrange elements yield sufficiently accurate results in displacement as well as the stress results are acceptable in most common sense and, as a particular situation, for degree p=3 Serendipity and Lagrange elements converge desired results with relatively fine mesh. Further, Hierarchic element is compatible with the problem including non-curved boundary, enables reasonably accurate displacement and stress results, and is more efficient, in terms of conditioning, than Lagrange and Serendipity elements.

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