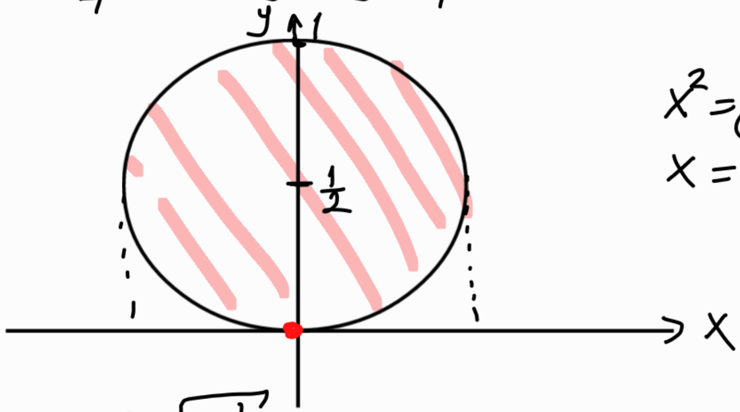
Ex.: Find the volume of the solid region R bounded by the paraboloid $z=1-x^2-y^2$ and below by the plane $z=1-y$.

Equating z-values: $\cancel{1}-x^2-y^2=\cancel{1}-y \Rightarrow \underset{-\frac{1}{4}}{x^2}=\underset{-\frac{1}{4}}{y}-y^2$

$$x^2-\frac{1}{4}=-\left(y^2-y+\frac{1}{4}\right)=-\left(y-\frac{1}{2}\right)^2 \Rightarrow x^2+\left(y-\frac{1}{2}\right)^2=\underset{r^2}{\frac{1}{4}}$$



$x^2=y-y^2$

$x=\mp\sqrt{y-y^2}$

$$V=\int_0^1\int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}}\Big[\underbrace{(1-x^2-y^2)-(1-y)}_{(-x^2-y^2+y)}\Big]\,dx\,dy=\int_0^1\left[-\frac{x^3}{3}-xy^2+xy\right]_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}}dy$$

$$=\int_0^1\left[\left(-\frac{(y-y^2)^{3/2}}{3}-y^2\cdot(y-y^2)^{1/2}+y\cdot(y-y^2)^{1/2}\right)-\left(\frac{(y-y^2)^{3/2}}{3}+y^2\cdot(y-y^2)^{1/2}-y\sqrt{y-y^2}\right)\right]dy$$

$$=\int_0^1\Big[\underbrace{-\frac{2}{3}(y-y^2)^{3/2}+\underbrace{(-2y^2+2y)}_{2(y-y^2)}\cdot(y-y^2)^{1/2}}_{2\cdot(y-y^2)^{3/2}}\Big]dy=\int_0^1\frac{4}{3}(y-y^2)^{3/2}\,dy$$

$$y-y^2=-\Big(\underbrace{y^2-y+\frac{1}{4}}_{-\left(y-\frac{1}{2}\right)^2+\frac{1}{4}}-\frac{1}{4}\Big)$$

$$=\frac{4}{3}\int_0^1\left[\frac{1}{4}-\left(y-\frac{1}{2}\right)^2\right]^{3/2}dy$$

$y-\frac{1}{2}=\frac{1}{2}\cos t \Rightarrow dy=-\frac{1}{2}\sin t\,dt$

$y=0\Rightarrow t=\pi \qquad y=1\Rightarrow t=0$

$$=\frac{4}{3}\int_\pi^0\Big[\underbrace{\frac{1}{4}-\frac{1}{4}\cos^2 t}_{\frac{1}{4}\cdot(1-\cos^2 t)=\frac{1}{4}\sin^2 t}\Big]^{3/2}\cdot\left(-\frac{1}{2}\right)\cdot\sin t\,dt=\frac{4}{3}\int_\pi^0\frac{1}{8}\cdot\sin^3 t\cdot\left(-\frac{1}{2}\right)\cdot\sin t\,dt$$

$$= \frac{4}{3}\cdot\frac{1}{8}\cdot\left(+\frac{1}{2}\right)\int_0^{\pi}\sin^4 t\,dt = \frac{1}{12}\int_0^{\pi}\underbrace{\left(\frac{1-\cos(2t)}{2}\right)^2}\,dt$$

$\sin^2 t = \frac{1-\cos(2t)}{2}$

$$\frac{1}{4}\left(1-\cos(2t)+\underbrace{\cos^2(2t)}_{\frac{1+\cos(4t)}{2}}\right)dt$$

$$= \frac{\pi}{32}$$

Ex.: $\iint_R \frac{y}{x^2+1}\,dA$, $R=\{(x,y) \mid 0\leq x\leq 4,\ 0\leq y\leq \sqrt{x}\}$

$y=0$ $\qquad y=\sqrt{x}$

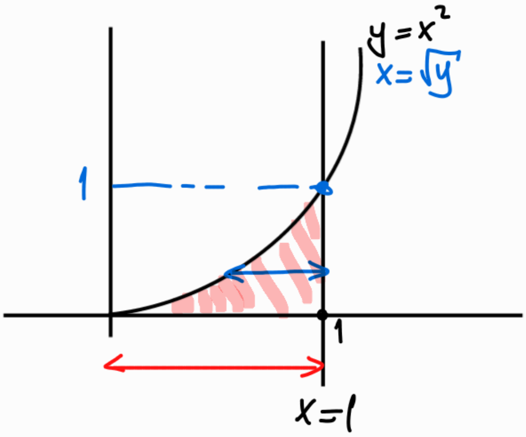
$$\int_0^4\int_0^{\sqrt{x}} \frac{y}{x^2+1}\,dy\,dx = \int_0^4 \frac{y^2}{2(x^2+1)}\Big|_0^{\sqrt{x}}\,dx = \int_0^4 \frac{1}{2(x^2+1)}\cdot x\,dx$$

$2x^2+2=t$
$4x\,dx=dt$
$x\,dx=\frac{dt}{4}$
$x=0 \Rightarrow t=2$
$x=4 \Rightarrow t=34$

$$= \int_2^{34}\frac{1}{t}\cdot\frac{dt}{4} = \frac{1}{4}\ln t\Big|_2^{34} = \frac{1}{4}(\ln 34-\ln 2) = \frac{1}{4}\ln 17$$

Ex.: $\iint_R x\cos y\,dA$, R is bounded by $y=0$, $y=x^2$, $x=1$

$\downarrow$
x-axis



$$I = \int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 x\sin y\Big|_0^{x^2}dx$$

$$= \int_0^1 x\sin x^2\,dx \qquad x^2=t \Rightarrow 2x\,dx=dt$$

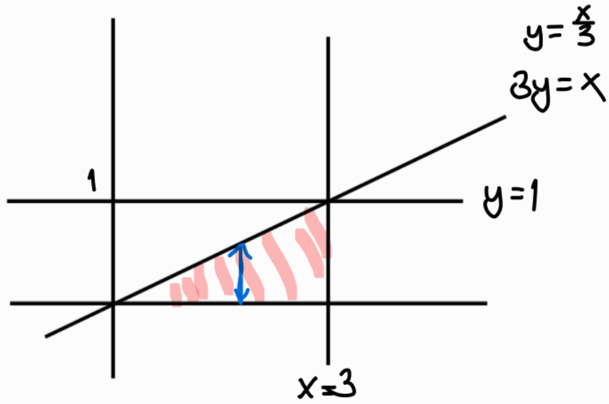
$x=0 \Rightarrow t=0$, $x=1 \Rightarrow t=1$

$$= \int_0^1 \sin t\,\frac{dt}{2} = -\frac{\cos t}{2}\Big|_0^1 = \frac{1-\cos 1}{2}$$

$$I = \int_0^1\int_{\sqrt{y}}^1 x\cos y\,dx\,dy$$

$\underline{Ex.:}$ $\displaystyle\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy$

x-axis
$0 \le y \le 1$
$3y \le x \le 3$



$$\int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 e^{x^2}\cdot y\,\Big|_0^{x/3}\,dx$$

$$= \int_0^3 e^{x^2}\cdot\frac{x}{3}\,dx$$

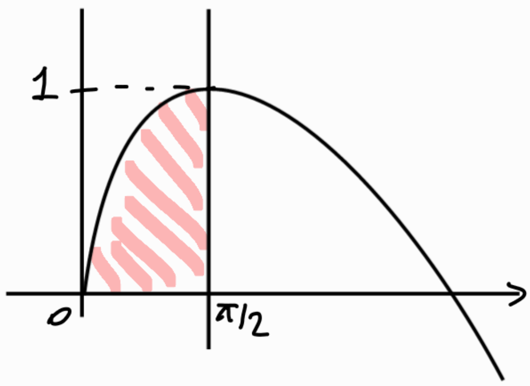
$x^2=t \Rightarrow 2x\,dx = dt$
$x=0 \Rightarrow t=0,\ x=3 \Rightarrow t=9$

$$= \frac{1}{3}\int_0^9 e^t\,\frac{dt}{2} = \frac{1}{6}\,e^t\,\Big|_0^9$$

$$= \frac{1}{6}\left(e^9-1\right)$$

$\underline{Ex.:}$ $\displaystyle\int_0^1\int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy$

$0 \le y \le 1$
$\arcsin y \le x \le \frac{\pi}{2}$



$x=\arcsin y \iff y=\sin x$

$$I = \int_0^{\pi/2}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dy\,dx$$

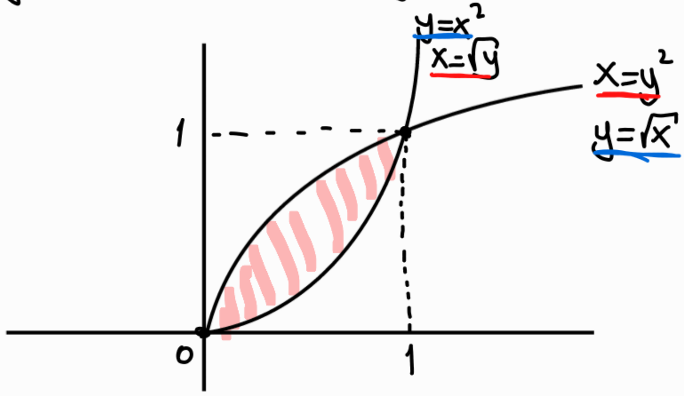
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\cdot y\,\Big|_0^{\sin x}\,dx$$

$$= \int_0^{\pi/2} \cos x\cdot\sin x\sqrt{1+\cos^2 x}\,dx$$

$\cos x = t \Rightarrow -\sin x\,dx = dt$
$x=0 \Rightarrow t=1 \ /\ x=\frac{\pi}{2} \Rightarrow t=0$

$$= \int_1^0 -t\sqrt{1+t^2}\,dt = -\frac{\cancel{2}}{3}\,\frac{\left(1+t^2\right)^{3/2}}{\cancel{2}}\,\Big|_1^0 = \frac{1}{3}\left(2\sqrt{2}-1\right)$$

Ex.: Find the volume of the solid under the plane $3x+2y-z=0$ and above the region enclosed by the parabolas $y=x^2$ and $x=y^2$



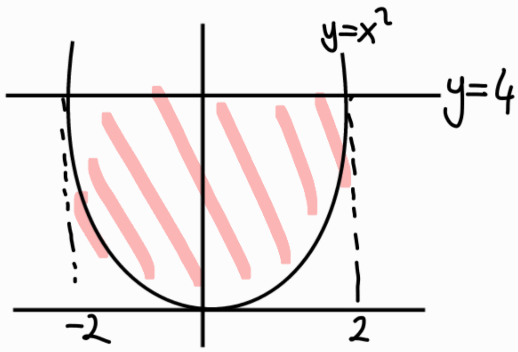
$$3x+2y-z=0 \Rightarrow z=3x+2y$$

$$V=\int_0^1\int_{x^2}^{\sqrt{x}}(3x+2y)\,dy\,dx=\frac{3}{4}$$

$$\int_0^1\int_{y^2}^{\sqrt{y}}(3x+2y)\,dx\,dy$$

Ex.: Find the volume of the solid enclosed by the cylinders $z=x^2$, $y=x^2$ and the planes $z=0$, $y=4$

$$\int_{-2}^{2}\int_{x^2}^{4}(x^2-0)\,dy\,dx=\frac{128}{15}$$



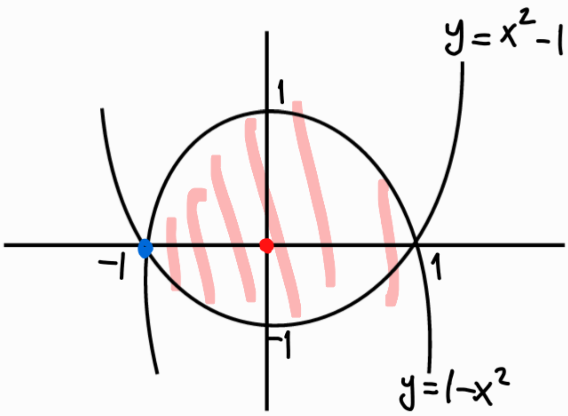
Ex.: Find the volume of the solid enclosed by the parabolic cylinders $y=1-x^2$, $y=x^2-1$ and the planes $x+y+z=2$, $2x+2y-z+10=0$.

$(x,y)=(0,0) \Rightarrow z=2$ (below) $(x,y)=(0,0)\Rightarrow z=10$ (above)

$\boxed{z=2-x-y}$ $\boxed{z=2x+2y+10}$

$$\int_{-1}^{1}\int_{x^2-1}^{1-x^2}\left[(2x+2y+10)-(2-x-y)\right]dy\,dx$$

$$=\frac{64}{3}$$



H.W: Find the volume of the solid under the plane $z=3$, above the plane $z=y$, and between the parabolic cylinders $y=x^2$ and $y=1-x^2$

$\left(\frac{10}{3\sqrt{2}} \ \underline{\underline{OR}} \ \frac{5\sqrt{2}}{3} \right)$