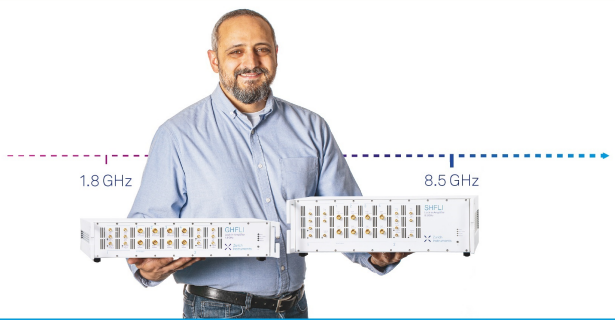
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Erdal Gül

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On A Regularized Trace Formula

Erdal Gül

Department of Mathematics, Faculty of Arts and Sciences, Yıldız Technical University, 34220 Istanbul, Turkey.

Corresponding author: gul@yildiz.edu.tr

Abstract. In this paper, we study the spectral properties of a self-adjoint differential operator with bounded operator-valued coefficient defined on a separable Hilbert space and derive a regularized trace formula for this operator.
Keywords: Hilbert space, spectrum, resolvent, regularized trace.
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INTRODUCTION

Let $\mathscr{H}$ be a separable Hilbert space. Consider a linear operator A whose domain $D(A)$ is dense in $\mathscr{H}$ (A is called densely defined operator in $\mathscr{H}$) and the inner product (Au,v) for a given fix v and every u in $D(A)$. Then there exists an element v^* for which

$$(Au,v)=(u,v^*)$$

holds. In the representation $v^*=A^*v$, A^* is called the adjoint operator of A. By assumption, since $D(A)$ is dense in $\mathscr{H}$ the element v^* is uniquely determined by the element v. We readily observe the following properties.

(a) A^* is a linear operator.
(b) If $A\subseteq B$, then $B^*\subseteq A^*$.
(c) The operator A^* is closed even when A is not closed.
(d) If A has a closure $\overline{A}$, then $(\overline{A})^*=A^*$.
(e) If the operator A^{**} exists, then $A\subseteq A^{**}$.

A linear operator A defined on $D(A)$ is called symmetric, if for every $u,v\in D(A)$, the equality

$$(Au,v)=(u,Av)$$

holds. A symmetric operator A has an extension as $A^*\supseteq A$.

A linear operator A defined on $D(A)$ is called self-adjoint, if $A=A^*$. From this definition we see that every self-adjoint operator is symmetric. For the bounded operators these two notions are equivalent.

Knowing that a number λ is a regular point of an operator A if the operator $R(\lambda)=(A-\lambda I)^{-1}$ exists, is defined on $\mathscr{H}$ and is bounded. The spectrum of A is set of all non-regular points and its discrete spectrum consists of the set of all eigenvalues. Especially, the discrete spectrum of a self-adjoint operator is a countable set of real numbers. The reader can find more details about the theory in the references [1,2].

Let $\mathscr{H}_1=L_2([0,\pi];\mathscr{H})$ represent the set of all measurable functions f defined on $[0,\pi]$ with their values in $\mathscr{H}$ such that for every $g\in\mathscr{H}$ the scalar function $(f(t),g)$ is measurable in the finite interval $[0,\pi]$ and

$$\int_0^\pi \|f(t)\|^2dx<\infty.$$

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Define the operators L_0 and L on $\mathscr{H}_1$ by:

$$L_0(y) = -y''(t) \ ; \quad y(0) = y(\pi), y'(0) = y'(\pi),$$

$$L(y) = -y''(t) + Q(t)y(t) \ ; \quad y(0) = y(\pi), y'(0) = y'(\pi).$$

Assume that the operator valued function $Q(t)$ has the following properties:

1. $Q(t)$ has a weak second-order derivative in $[0,\pi]$ and for $t \in [0,\pi]$, $Q^{(i)}(t)$ $(i = 0,1,2)$ is a self adjoint kernel operator on $\mathscr{H}$.
2. $\|Q\|_{\mathscr{H}_1} < 2$.
3. $\mathscr{H}_1$ has an o.n.b. $\{\varphi_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|Q\varphi_n\|_{\mathscr{H}_1} < \infty$.
4. $\left\|Q^{(i)}(t)\right\|_{S_1[\mathscr{H}]}$ $(i = 0,1,2)$ is a bounded measurable function on $[0,\pi]$.

In [3,4,5,6,7,8,9], some trace formulas are obtained for different types of differential operators. In this study, we compute a regularized trace formula for the self-adjoint operator L through the similar techniques used in [10].

MAIN RESULTS

The spectrum of L_0 is $\sigma(L_0) = \{(2m)^2\}_{m=0}^{\infty}$ and each $\lambda \in \sigma(L_0)$ is an eigenvalue of infinite multiplicity for L_0 . Let $R_0(\lambda) = (L_0 - \lambda I)^{-1}$, $R(\lambda) = (L - \lambda I)^{-1}$ be the resolvents of the operators L_0 and L , respectively.

Lemma 1 *If $\lambda \notin \sigma(L_0)$ then $QR_0 \in \mathbb{S}_1[\mathscr{H}_1]$, that is, $QR_0 : \mathscr{H}_1 \longrightarrow \mathscr{H}_1$ is a kernel operator.*

Theorem 1 *If the conditions (2) and (3) on $Q(t)$ are satisfied then the spectrum of L, $\sigma(L)$, is a subset of the union of pairwise disjoint intervals $I_m = [4m^2 - \|Q\|_{\mathscr{H}_1}, 4m^2 + \|Q\|_{\mathscr{H}_1}]$ $(m = 0,1,2,...)$ and*

(i) Each point of $\sigma(L)$ which is not equal to $4m^2$ is an isolated eigenvalue of finite multiplicity.
(ii) $k_m = 4m^2$ itself can be an eigenvalue of L with either finite or infinite multiplicity.
(iii)

$$\lim_{n\to\infty} \lambda_{mn} = k_m$$

where $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of L belonging to I_m.

Lemma 2 *The operator-valued function $R - R_0$ is analytic in the resolvent set of L, with respect to the $\mathbb{S}_1[\mathscr{H}_1]$ norm.*

Let $\{\psi_{mn}(t)\}_{m=0,n=1}^{\infty}$ be orthonormal eigenfunctions corresponding to eigenvalues $\{\lambda_{mn}\}_{m=0,n=1}^{\infty}$ of L and

$$B^{\circ}_{mn} = (.,\psi^{\circ}_{mn})_{\mathscr{H}_1}\psi^{\circ}_{mn};\ B_{mn} = (.,\psi_{mn})_{\mathscr{H}_1}\psi_{mn}.$$

Thus we have

$$R_0(\lambda) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{B^{\circ}_{mn}}{k_m - \lambda};\ R(\lambda) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{B_{mn}}{\lambda_{mn} - \lambda} \tag{1}$$

and

$$tr(R(\lambda) - R_0(\lambda)) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \Big(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{k_m - \lambda}\Big).$$

since $R - R_0 \in \mathbb{S}_1[\mathscr{H}_1]$, for every $\lambda \in \rho(L)$.

If multiply both sides of this last equality by $\lambda/2\pi i$ and integrate over the circle $|\lambda| = b_p = k_p + p$ $(p = 1,2,\ldots)$ we find:

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty} (\lambda_{mn} - k_m) = \sum_{j=1}^{N} M_j^p + M_{pN} \tag{2}$$

where

$$M_j^p = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p} \lambda\, tr\big[(QR_0^j(\lambda)\big]d\lambda \tag{3}$$

and

$$M_{pN} = \frac{(-1)^N}{2\pi i}\int_{|\lambda|=b_p} \lambda\, tr\big[R(\lambda)(QR_0^{N+1}(\lambda))\big]\, d\lambda. \tag{4}$$

for a natural N. Now, it is not difficult to see that

$$M_j^p = \frac{(-1)^j}{2\pi i j}\int_{|\lambda|=b_p} tr\big[(QR_0^j(\lambda)\big]d\lambda. \tag{5}$$

Lemma 3 *With the conditions* (1), (2), (3) *on* $Q(t)$ *we have*

$$M_1^p = \frac{f(p)}{\pi}\int_0^\pi trQ(t)\,dt.$$

Theorem 2 *If the conditions* (1), (2), (3) *on* $Q(t)$ *are satisfied then we have*

$$\lim_{p\to\infty} M_j^p = 0 \quad (j = 2,3,4,\ldots).$$

Theorem 3 *If the conditions* $(1)-(4)$ *on* $Q(t)$ *are satisfied then the following equation holds:*

$$\lim_{p\to\infty}\left\{\sum_{m=0}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}-k_m^2)+\sum_{m=1}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}-k_m^2)-\frac{f(p)}{\pi}\int_0^\pi trQ(t)\,dt\right\}=0 \tag{6}$$

The equation (6) is called the regularized trace formula of the operator L.

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