END4400 – System Dynamics

Week 4 – 30/3/2021

Assist. Prof. Mert Edalı
medali@yildiz.edu.tr

Outline

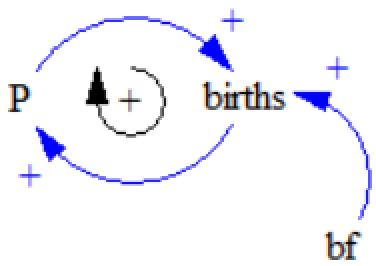
- Basic Stock – Flow Dynamics
- Simple Positive and Negative Feedback Loops, Doubling Time/Half Life

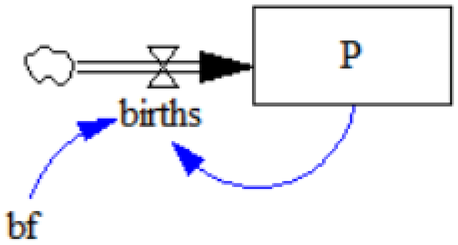
Basic Stock – Flow Dynamics

- consider an SD model with a positive feedback loop
- assume that *births* are proportional to the population (*P*)

$$births = P * bf$$

  - *bf*: birth fraction



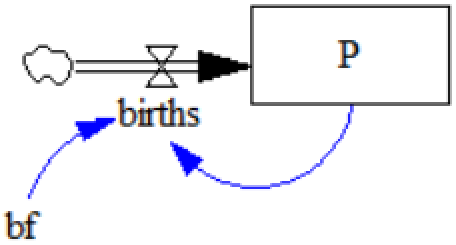


- units:
  - *P*: people
  - *births*: people/time
  - *bf*: 1/time

Basic Stock – Flow Dynamics

- write down the differential equation

$$\frac{dP}{dt} = births$$



- write down the integral equation

$$P(t) = P(0) + \int_0^t births(s)ds$$

- plug in all variables

$$\frac{dP}{dt} = P * bf$$

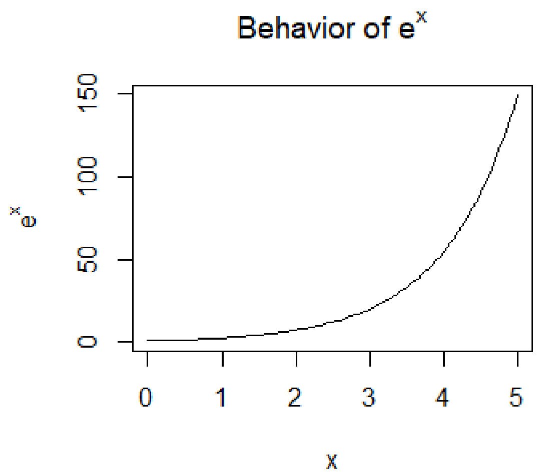
$$\frac{dP}{P} = bf * dt$$

$$\int \frac{dP}{P} = \int bf * dt$$

$$lnP = bf * t + C$$

$$P(t) = e^{bf*t+C} = e^{bf*t}e^{C} = e^{bf*t}C$$

behavior?



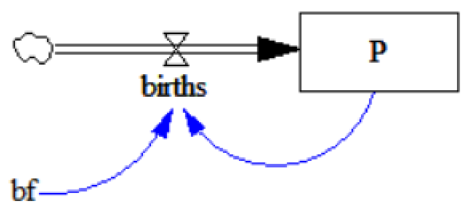
Basic Stock – Flow Dynamics

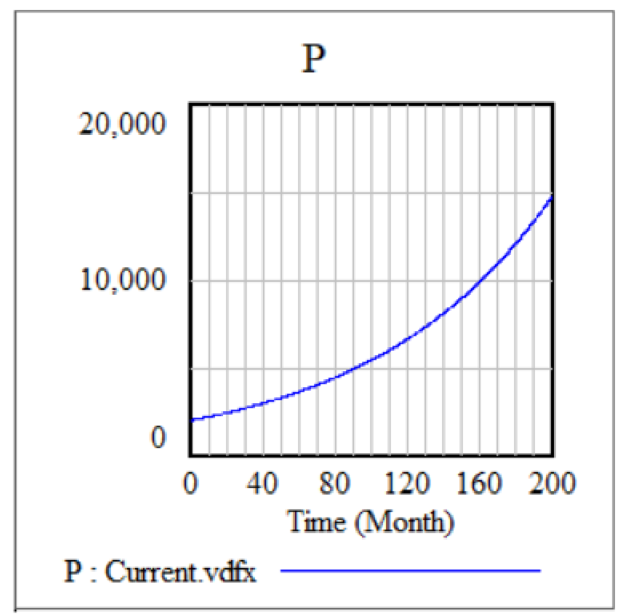
- assume
  - P(0) = 2000 (people)
  - bf = 0.01 (1/time)

$$P(0) = e^{0.01*0}C = 2000$$

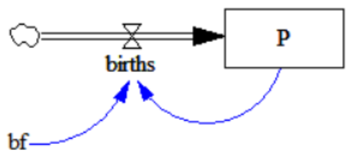
$$C = 2000$$

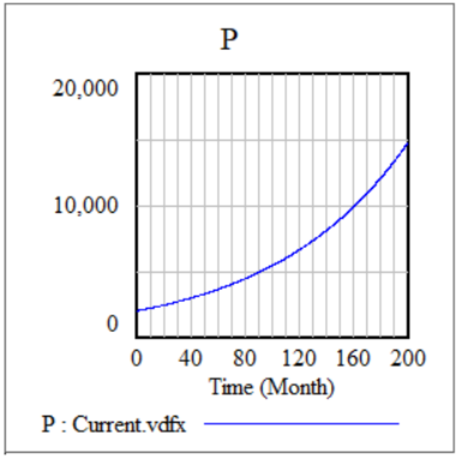
$$P(t) = 2000 * e^{0.01*t}$$





Basic Stock – Flow Dynamics

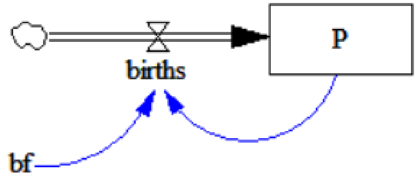


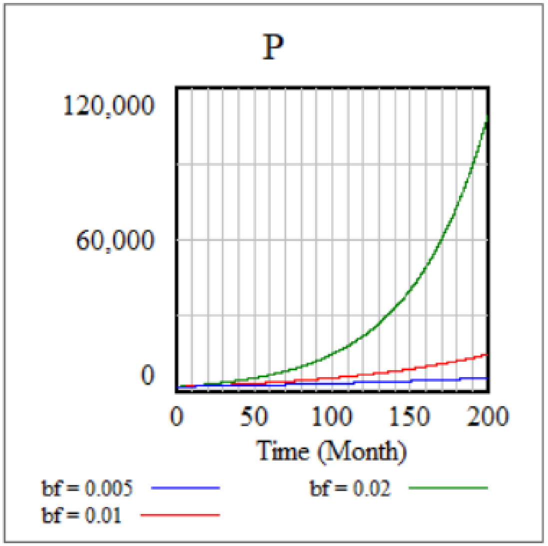


- behavior → exponential growth
  - positive (reinforcing) feedback loop
  - The greater the state of the system, the greater the inflow: this is precisely the meaning of the positive feedback loop coupling the stock and its inflow.
- how does the behavior change with respect to different values of *bf*?

Basic Stock – Flow Dynamics

- as *bf* increases, population (P) grows faster





Basic Stock – Flow Dynamics

- what kind of behavior do you expect to see when we have *births* on the *x*-axis and *P* on the *y*-axis?
  - reconsider the differential equation again;

$$\frac{dP}{dt} = P * bf$$

→ the rate of change (inflow) is a linear function of the state of the system

explains how the state of the system changes over time

  - answer will be given next week! try to guess

Basic Stock – Flow Dynamics

- Positive feedback loops are the most powerful processes in the universe.
  - the rate of increase (inflow) grows as the state of the system (stock/state variable) grows.
- an important property of exponential growth (and thus, positive feedback loop)
  - the state of the system doubles in a fixed period of time
    - doubling time
  - e.g., assume doubling time is 3 days
  - it takes 3 days for the state going from 6 units to 12 units
  - it takes 3 days for the state going from 2500 units to 5000 units

Basic Stock – Flow Dynamics

- how to calculate the doubling time for P(0) = 2000 (people) and bf = 0.01 (1/time);
  - the solution is $P(t) = 2000 * e^{0.01*t}$
  - the presence of doubling time means that the value of stock will double itself in a certain period of time (t_d)

$$\frac{2000 * e^{0.01*t_2}}{2000 * e^{0.01*t_1}} = 2$$

$$t_2 > t_1$$

$$e^{0.01*(t_2 - t_1)} = 2$$

$$ln2 = 0.01 * (t_2 - t_1)$$

$$\frac{ln2}{0.01} = (t_2 - t_1) = t_d$$

$$t_d = 69.3147$$

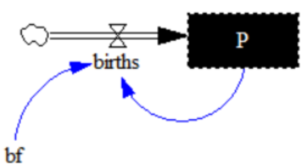


Table Time Down

Time (Month	"P" Runs:	P
68.9375	Current	3984.08
69	Current	3986.57
69.0625	Current	3989.06
69.125	Current	3991.56
69.1875	Current	3994.05
69.25	Current	3996.55
69.3125	Current	3999.05
69.375	Current	4001.54
69.4375	Current	4004.05
69.5	Current	4006.55
69.5625	Current	4009.05
69.625	Current	4011.56
69.6875	Current	4014.07
69.75	Current	4016.57
69.8125	Current	4019.08
69.875	Current	4021.6
69.9375	Current	4024.11

Basic Stock – Flow Dynamics

- t_d is valid for each value;

Table Time Down

Time (Month	"P" Runs:	P
22.0625	Current	2493.54
22.125	Current	2495.1
22.1875		2496.66
22.25		2498.22
22.3125		2499.78
22.375		2501.34
22.4375		2502.91
22.5		2504.47
22.5625		2506.03
22.625		2507.6
22.6875		2509.17
22.75		2510.74
22.8125		2512.31
22.875		2513.88
22.9375		2515.45
23		2517.02
23.0625		2518.59

Table Time Down

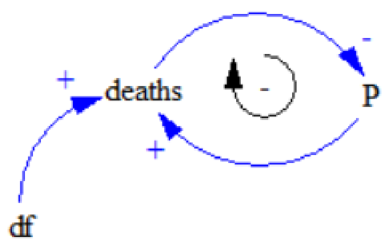
Time (Month	"P" Runs:	P
90.875	Current	4961.03
90.9375		4964.13
91		4967.23
91.0625		4970.34
91.125		4973.44
91.1875		4976.55
91.25		4979.66
91.3125		4982.77
91.375		4985.89
91.4375		4989.01
91.5		4992.12
91.5625		4995.24
91.625		4998.37
91.6875		5001.49
91.75		5004.62
91.8125		5007.74
91.875		5010.87

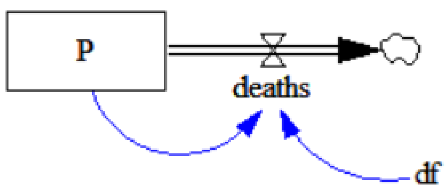
- pick a stock value and record its time
- add t_d to that recorded value
- you will see that the stock doubled itself

$22.375 + 69.3147 = 91.6897$

Basic Stock – Flow Dynamics

- consider an SD model with a negative feedback loop
- assume that *deaths* are proportional to the population (*P*)
  - *df*: death fraction



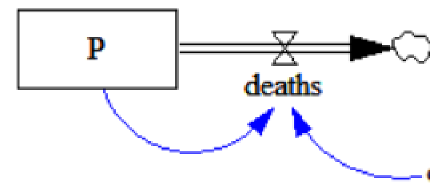


- units:
  - *P*: people
  - *deaths*: people/time
  - *df*: 1/time

Basic Stock – Flow Dynamics

- write down the differential equation

$$\frac{dP}{dt} = -deaths$$



- write down the integral equation

$$P(t) = P(0) + \int_0^t -deaths(s)ds$$

- plug in all variables

$$\frac{dP}{dt} = -P * df$$

↓

$$\frac{dP}{dt} = -P * df$$

↓

$$\int \frac{dP}{P} = \int -df * dt$$

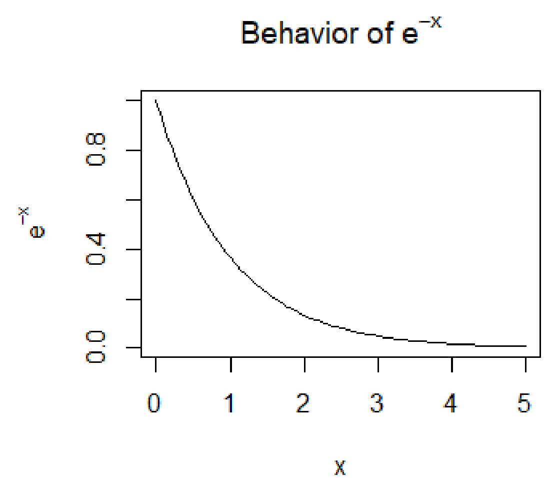
$$lnP = -df * t + C$$

↓

$$P(t) = e^{-df*t+C} = e^{-df*t}e^{C} = e^{-df*t}C$$

↓

behavior?



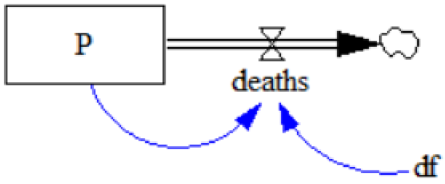
Basic Stock – Flow Dynamics

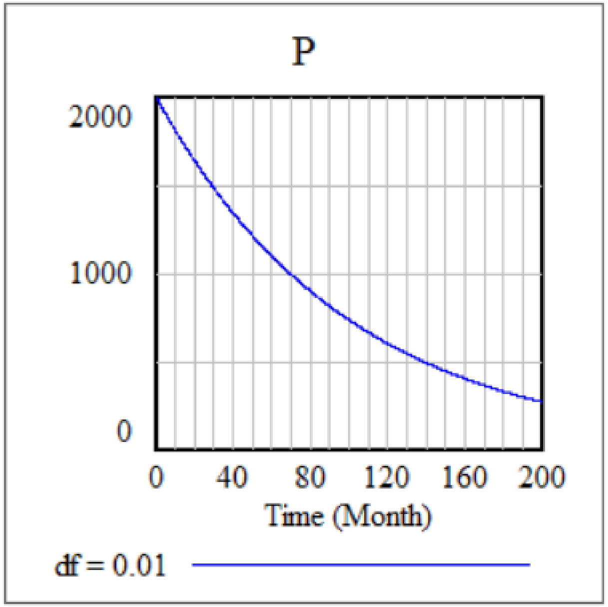
- assume
  - P(0) = 2000 (people)
  - df = 0.01 (1/time)

$$P(0) = e^{-0.01*0}C = 2000$$

$$C = 2000$$

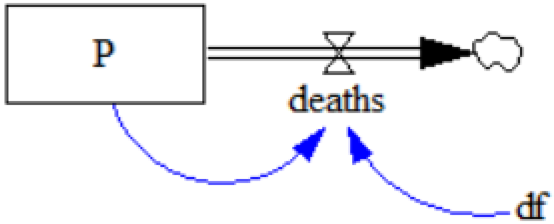
$$P(t) = 2000 * e^{-0.01*t}$$

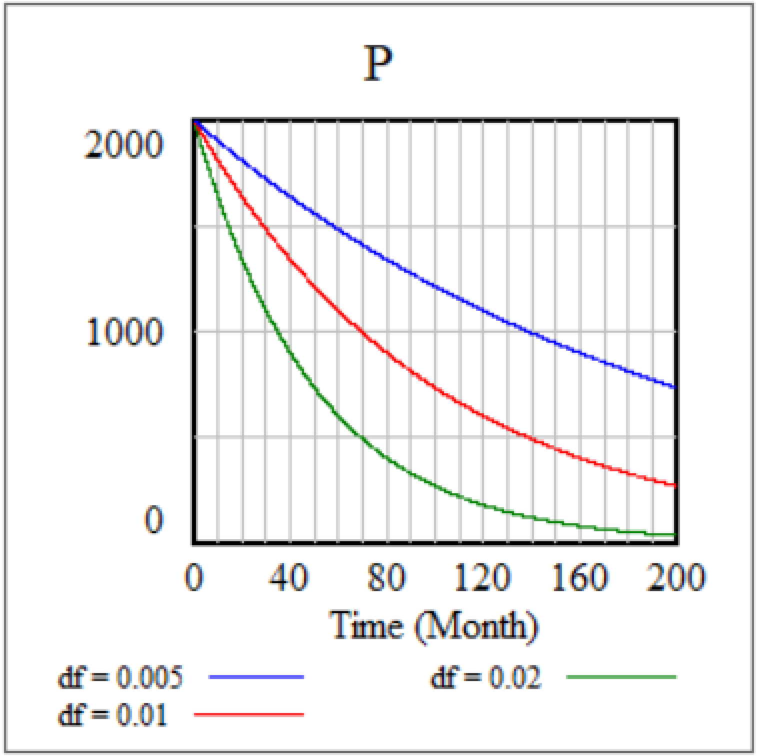




Basic Stock – Flow Dynamics

- as *df* increases, population (P) drops faster





Basic Stock – Flow Dynamics

- negative feedback loops exhibit exponential decay
- the state variable has a half-life
  - the stock halves itself in a certain period of time

Basic Stock – Flow Dynamics

- how to calculate the half life for P(0) = 2000 (people) and df = 0.01 (1/time);
  - the solution is $P(t) = 2000 * e^{-0.01*t}$
  - the presence of half life means that the value of stock will halve itself in a certain period of time (t_h)

$$\frac{2000 * e^{-0.01*t_2}}{2000 * e^{-0.01*t_1}} = \frac{1}{2}$$

$$t_2 > t_1$$

$$e^{-0.01*(t_2-t_1)} = 0.5$$

$$ln0.5 = -0.01 * (t_2 - t_1)$$

$$\frac{ln0.5}{-0.01} = (t_2 - t_1) = t_h$$

$$t_h = 69.3147$$

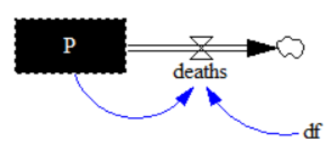
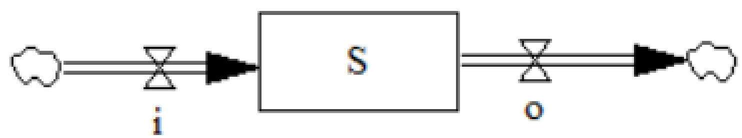


Table Time Down

Time (Month	"P" Runs: P
68.5625	1007.33
68.625	1006.71
68.6875	1006.08
68.75	1005.45
68.8125	1004.82
68.875	1004.19
68.9375	1003.56
69	1002.94
69.0625	1002.31
69.125	1001.68
69.1875	1001.06
69.25	1000.43
69.3125	999.806
69.375	999.181
69.4375	998.556
69.5	997.932
69.5625	997.308

Basic Stock – Flow Dynamics

- all the previous differential equations are analytically solvable
- most SD models with moderate realism, analytical solutions cannot be obtained
- solution → numerical integration



stock-flow diagram

$$S(t) = S(0) + \int_0^t \big(i(s) - o(s)\big)ds$$

integral equation

not always possible to integrate

Basic Stock – Flow Dynamics

- instead of solving the integral (or the differential equation), it is possible to perform numerical integration;

  $S(t + dt) = S(t) + \big(i(t) - o(t)\big) * dt$ → Euler's integration method

  approximate integral equation

- dt → (simulation) time step
  - should be "small enough" for accuracy
- As the time step (dt) shrinks, the accuracy of Euler's approximation improves.
- Vensim simply performs numerical integration for simulation.

https://en.wikipedia.org/wiki/Numerical_integration

Basic Stock – Flow Dynamics

- let us take a look at the approximate integral equation one more time;

$$S(t + dt) = S(t) + \big(i(t) - o(t)\big) * dt$$

$$\frac{S(t + dt) - S(t)}{dt} = \big(i(t) - o(t)\big)$$

$$\lim_{dt \to 0} \frac{S(t + dt) - S(t)}{dt} = \frac{dS}{dt}$$

definition of derivative

- the smaller the value of *dt* is, the better the quality of the approximation is.

Basic Stock – Flow Dynamics

- Since Vensim performs an approximation via numerical integration, there will always be an approximation error.
- consider the negative feedback loop example

$$P(t) = 2000 * e^{-0.01*t}$$

Time (Month)	Vensim dt = 1	Vensim dt = 0.5	Vensim dt = 0.125	Vensim dt = 0.0625	Analytical
0	20000	20000	20000	20000	20000
1	19800	19800.5	19800.87305	19800.93555	19801
2	19602	19602.99023	19603.72852	19603.85156	19603.97
3	19405.98047	19407.45117	19408.54688	19408.72852	19408.91
4	19211.91992	19213.86133	19215.30859	19215.54883	19215.79
5	19019.80078	19022.20313	19023.99414	19024.29102	19024.59
6	18829.60352	18832.45703	18834.58398	18834.9375	18835.29
7	18641.30664	18644.60156	18647.06055	18647.46875	18647.88
8	18454.89453	18458.62305	18461.40234	18461.86523	18462.33
9	18270.3457	18274.49805	18277.5957	18278.10938	18278.62
10	18087.64063	18092.20898	18095.61719	18096.18359	18096.75
11	17906.76563	17911.74023	17915.44922	17916.06641	17916.68
12	17727.69727	17733.07031	17737.07813	17737.74414	17738.41
13	17550.41992	17556.18359	17560.48047	17561.19531	17561.91
14	17374.91602	17381.06055	17385.64258	17386.4043	17387.16
15	17201.16797	17207.68359	17212.54492	17213.35156	17214.16

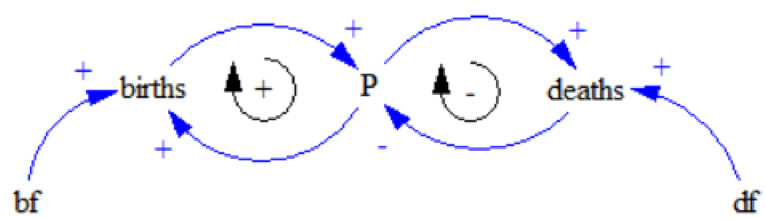
Time (Month)	Vensim dt = 1	Vensim dt = 0.5	Vensim dt = 0.125	Vensim dt = 0.0625
0				
1	-0.0051%	-0.0025%	-0.0006%	-0.0003%
2	-0.0100%	-0.0050%	-0.0012%	-0.0006%
3	-0.0151%	-0.0075%	-0.0019%	-0.0009%
4	-0.0201%	-0.0100%	-0.0025%	-0.0013%
5	-0.0252%	-0.0125%	-0.0031%	-0.0016%
6	-0.0302%	-0.0150%	-0.0037%	-0.0019%
7	-0.0352%	-0.0176%	-0.0044%	-0.0022%
8	-0.0403%	-0.0201%	-0.0050%	-0.0025%
9	-0.0453%	-0.0226%	-0.0056%	-0.0028%
10	-0.0503%	-0.0251%	-0.0063%	-0.0031%
11	-0.0553%	-0.0276%	-0.0069%	-0.0034%
12	-0.0604%	-0.0301%	-0.0075%	-0.0038%
13	-0.0654%	-0.0326%	-0.0081%	-0.0041%
14	-0.0704%	-0.0351%	-0.0087%	-0.0043%

percent errors with respect to the analytical solution

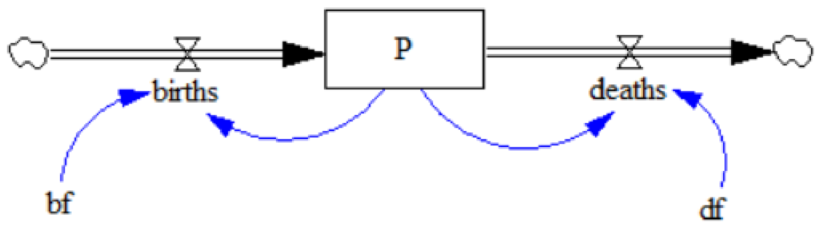
Basic Stock – Flow Dynamics

- coupling positive and negative feedback loops
  - population model with births and deaths

- causal loop diagram



- stock-flow diagram



Basic Stock – Flow Dynamics

- write down the differential equation

$$\frac{dP}{dt} = births - deaths$$

- write down the integral equation

$$P(t) = P(0) + \int_0^t \left(births(s) - deaths(s)\right) * ds$$

- write down the approximate integral equation

$$P(t + dt) = P(t) + \underbrace{[births(t) - deaths(t)]}_{\text{net flow}} * dt$$

Basic Stock – Flow Dynamics

- solve the differential equation

$$\frac{dP}{dt} = P * (bf - df)$$

$$\frac{dP}{P} = (bf - df) * dt$$

$$\int \frac{dP}{P} = \int (bf - df) * dt$$

$$lnP = (bf - df) * t + C$$

$$P(t) = e^{(bf-df)*t+C} = e^{(bf-df)*t}e^{C} = e^{(bf-df)*t}C$$

assume

P(0) = 3000

bf = 0.03

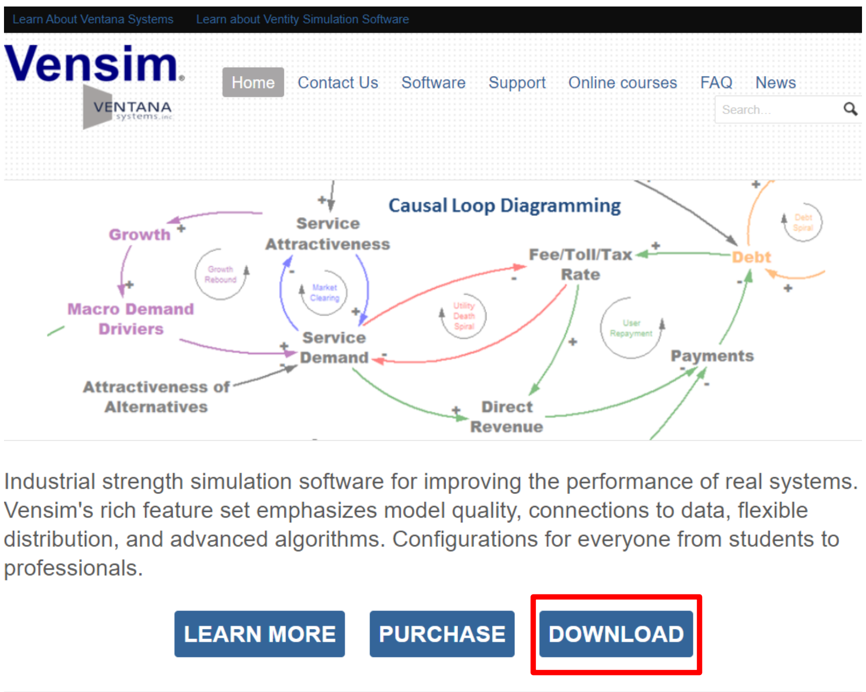
df = 0.01

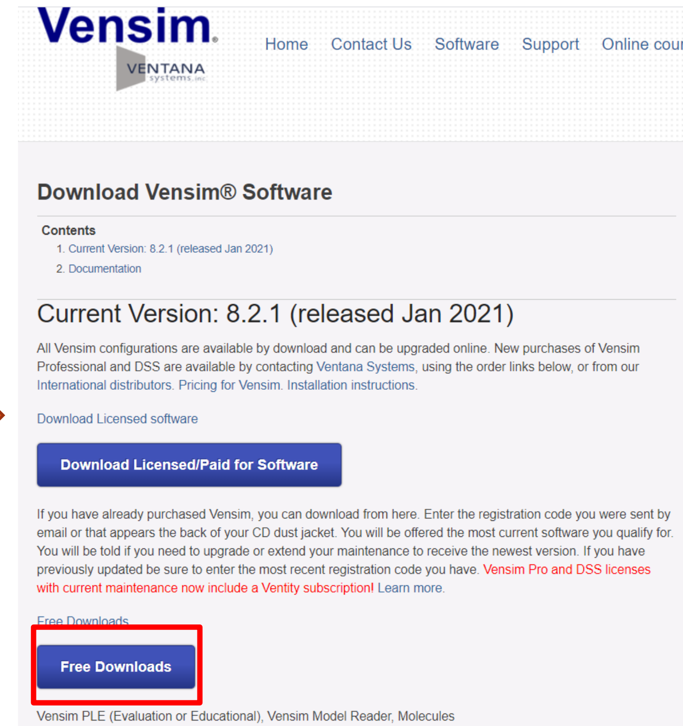
$$P(t) = 3000 * e^{0.02*t}$$

exponential growth

Basic Stock – Flow Dynamics

- Homework:
  - download and setup Vensim (vensim.com)





Basic Stock – Flow Dynamics

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Basic Stock – Flow Dynamics

- References

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Sterman, J. Business Dynamics. Systems Thinking and Modeling for a Complex World. McGraw-Hill, U.S.A., 2000.