Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/070410

The Taylor Matrix Method for Approximate Solution of Lane-Emden Equation with index-n

*Coskun Guler**, *Ibrahim Emiroglu, Fatih Tasci and Mustafa Sivri*

Yildiz Technical University, Mathematical Engineering Department, Davutpasa Campus, 34210, Esenler, Istanbul, Turkey.

Received: 13 Jan. 2013, Revised: 1 Mar. 2013, Accepted: 28 Mar. 2013
Published online: 1 Jul. 2013

Abstract: Many problems in mathematical physics can be formulated as an equation of Lane-Emden type. There are many methods for the solution of this equation. One of these methods is the Taylor matrix method. The only types of nonlinear equations that this method has been applied so far are the Riccati and Abel equations. In this study, an algorithm based on the Taylor matrix method is proposed and applied to the nonlinear Lane-Emden equation with index-n. An example is also given.

Keywords: Lane-Emden equation, the Taylor matrix method, Nonlinear differential equations.

1 Introduction

The nonlinear differential equations are indispensable tools for modeling many physical phenomenon such as chemical reactions, spring-mass system and bending of beams. These equations are also useful in ecology and economics [1,2,3]. Therefore, the solution methods for these equations have gained importance for engineers and scientists. The aim of this paper is to solve the Lane-Emden equation by making use of the Taylor matrix method [4,5,6,7,8]. The method is used to solve a wide class of algebraic, difference and partial differential equations.

All studies made so far show that the Taylor matrix method have been used only for the solution of linear ordinary differential equations, Riccatti differential equation [9] and Abel equation [10].

In this study Lane-Emden equation [11,12] is solved by the Taylor matrix method. First we consider the following equation

$$y'' + \frac{2}{x}y' + y^n = 0, 0 < x < \infty \qquad (1)$$

under the conditions

$y(0) = 1, \quad y'(0) = 0$. Where $n > 0$ is an integer.

$$y(0) = 1, \quad y'(0) = 0. \textit{Where } n > 0 \textit{ is an integer.} \qquad (2)$$

If we rewrite equation (1) as $xy'' + 2y' + xy^n = 0$ and denote the coefficients of the equation by A, B and C respectively. Then we have

$$Ay'' + By' + Cy^n = 0.$$

The solution of equation (1) can be expressed

$$y(x) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-c)^n, \qquad (3)$$

where, y(x) is the solution of Eq.1, N is the degree of the Taylor polynomial at $x = c$ and $y^{(n)}(c)$, $n = 0, 1, \ldots, N$ are the coefficients to be determined.

2 Analysis of the Taylor matrix method

We can put the series (3) in the matrix form

$$[y(x)] = \mathbf{XMY} \qquad (4)$$

where

$$\mathbf{X} = \left[1 \; (x-c) \; (x-c)^2 \cdots (x-c)^N\right]$$

$$\mathbf{M} = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1!} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N!} \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} y^0(c) \\ y^1(c) \\ y^2(c) \\ . \\ . \\ . \\ y^n(c) \end{bmatrix}.$$

* Corresponding author e-mail: cguler@yildiz.edu.tr

© 2013 NSP
Natural Sciences Publishing Cor.

Now we consider the term Ay'' of Eq.(1). It can be written as the truncated Taylor series expansions of degree N at $x=c$ in the form

$$Ay'' = \sum_{n=0}^{N} \frac{1}{n!} [A(x)y(x)]^{(n)}_{x=c} (x-c)^n. \tag{5}$$

We can write

$[Ay]^{(n)}_{x=c} = \sum_{m=0}^{n} \binom{n}{m} A^{(n-m)}(c)y^{(m)}(c)$ and if we substitute it in (5) we get

$$Ay'' = \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{1}{n!} \binom{n}{m} A^{(n-m)}(c)y^{(m)}(c)(x-c)^{(n)}$$

and its matrix form

$$\left[Ay''\right] = XAY \tag{6}$$

where

$$\mathbf{A}= \begin{bmatrix} 0 & 0 & \frac{A^{(0)}(c)}{0!0!} & \cdots & 0 \\ 0 & 0 & \frac{A^{(1)}(c)}{1!0!} & \frac{A^{(0)}(c)}{0!1!} \cdots & 0 \\ 0 & 0 & \frac{A^{(2)}(c)}{2!0!} & \frac{A^{(1)}(c)}{1!1!} \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \frac{A^{(N)}(c)}{N!0!} & \frac{A^{(N-1)}(c)}{(N-1)!1!} \cdots & \frac{A^{(2)}(c)}{2!(N-2)!} \end{bmatrix}.$$

By analogy we obtain

$$B(x)y'(x) = \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{1}{n!} \binom{n}{m} B^{(n-m)}(c)y^{(m+1)}(c)(x-c)^{(n)}, \tag{7}$$

$$C(x)Y_1(x) = \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{1}{n!} \binom{n}{m} C^{(n-m)}(c)Y_1^{(m)}(c)(x-c)^{(n)} \tag{8}$$

$$C(x)Y_2(x) = \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{1}{n!} \binom{n}{m} C^{(n-m)}(c)Y_2^{(m)}(c)(x-c)^{(n)} \tag{9}$$

.

.

.

$$C(x)\overline{Y}(x) = \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{1}{n!} \binom{n}{m} C^{(n-m)}(c)\overline{Y}^{(m)}(c)(x-c)^{(n)} \tag{10}$$

where

$$Y_1(x) = (y(x))^2, \qquad Y_1^{(m)}(c) = \sum_{i=0}^{m} \binom{m}{i} y^{(m-i)}(c)y^{(i)}(c) \tag{11}$$

$$Y_2(x) = (y(x))^3, \qquad Y_2^{(1)}(c) = 3(y(c))^2 y^{(1)}(c)$$

for $m \geq 2$

$$Y_2^{(m)}(c) = 3\sum_{i=0}^{m-1} \binom{m-1}{i} y^{(m-i)}(c)Y_1^{(i)}(c) \tag{12}$$

$$\overline{Y}(x) = (y(x))^n, \qquad \overline{Y}^{(1)}(c) = n(y(c))^{n-1}y^{(1)}(c)$$

for $m \geq 2$

$$\overline{Y}^{(m)}(c) = n\sum_{i=0}^{m-1} \binom{m-1}{i} y^{(m-i)}(c)Y_{n-2}^{(i)}(c) \tag{13}$$

where $\overline{Y}(x) = Y_{n-1}(x)$. And their matrix representations, respectively

$$\left[Ay''\right] = \mathbf{XAY} \tag{14}$$

$$\left[By'\right] = \mathbf{XBY} \tag{15}$$

$$[Cy^n] = \left[C\overline{Y}\right] = \mathbf{XC\overline{Y}} \tag{16}$$

where

$$\mathbf{B}= \begin{bmatrix} 0 & \frac{B_1^{(0)}(c)}{0!0!} & 0 & \cdots & 0 \\ 0 & \frac{B_1^{(1)}(c)}{1!0!} & \frac{B_1^{(0)}(c)}{0!1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{B_1^{(N-1)}(c)}{(N-1)!0!} & \frac{B_1^{(N-2)}(c)}{(N-2)!1!} & \cdots & \frac{B_1^{(0)}(c)}{0!(N-1)!} \\ 0 & \frac{B_1^{(N)}(c)}{N!0!} & \frac{B_1^{(N-1)}(c)}{(N-1)!1!} & \cdots & \frac{B_1^{(1)}(c)}{1!(N-1)!} \end{bmatrix}, Y= \begin{bmatrix} y^{(0)} \\ y^{(1)} \\ \vdots \\ y^{(N)} \end{bmatrix},$$

$$\mathbf{C}= \begin{bmatrix} \frac{C^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{C^{(1)}(c)}{1!0!} & \frac{C^{(0)}(c)}{0!1!} & 0 & \cdots & 0 \\ \frac{C^{(2)}(c)}{2!0!} & \frac{C^{(1)}(c)}{1!1!} & \frac{C^{(0)}(c)}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{C^{(N)}(c)}{N!0!} & \frac{C^{(N-1)}(c)}{(N-1)!1!} & \frac{C^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{C^{(0)}(c)}{0!N!} \end{bmatrix},$$

$$\overline{Y}= \begin{bmatrix} Y_1^{(0)} \\ Y_1^{(1)} \\ \vdots \\ Y_1^{(N)} \end{bmatrix}.$$

Substituting the matrix forms (6)-(13) into the equation $Ay'' + By' + Cy^n = 0$, then we have the matrix equation

$$(\mathbf{A}+\mathbf{B})\mathbf{Y} + \mathbf{C\overline{Y}} = \mathbf{O}\,. \tag{17}$$

The matrix Equation (17) is fundamental relation for Lane-Emden equation (1).

© 2013 NSP
Natural Sciences Publishing Cor.

3 Solution by the Taylor matrix method

Consider

$$(\mathbf{A}+\mathbf{B})\mathbf{Y}+\mathbf{C}\overline{\mathbf{Y}}=\mathbf{O} \tag{18}$$

and let

$$\mathbf{A}+\mathbf{B}=\mathbf{U}=[u_{ij}],\mathbf{C}=[c_{ij}],i,\ j=0,1,...,N. \tag{19}$$

The augmented matrix of Equation (19) becomes

$$\mathbf{Q}=[\mathbf{U};\mathbf{C};\mathbf{O}] \tag{20}$$

where

$$\mathbf{U}=\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ u_{N0} & u_{N1} & \cdots & u_{NN} \end{bmatrix}, \quad \mathbf{C}=\begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0N} \\ c_{10} & c_{11} & \cdots & c_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N0} & c_{N1} & \cdots & c_{NN} \end{bmatrix},$$

$$\mathbf{O}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In order to find the unknown Taylor coefficients, replacing last two rows of matrices U, C and O with the proper rows is necessary to satisfy the initial conditions given by Eq.(2). Then, we have matrices

$$\mathbf{U}^*=\begin{bmatrix} u_{00} & u_{01} & \ldots & u_{0N} \\ u_{10} & u_{11} & \ldots & u_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ u_{N-20} & u_{N-21} & u_{N-22} & u_{N-2N} \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \end{bmatrix},$$

$$\mathbf{C}^*=\begin{bmatrix} c_{00} & c_{01} & \ldots & c_{0N} \\ c_{10} & c_{11} & \ldots & c_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N-20} & c_{N-21} & c_{N-22} & c_{N-2N} \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \mathbf{O}^*=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then, the corresponding matrix equation is

$$\mathbf{U}^*\mathbf{Y}+\mathbf{C}^*\overline{\mathbf{Y}}=\mathbf{O}^* \tag{21}$$

From the system (21), the unknown Taylor coefficients $y^{(n)}(c)$ $(n=0,1,\cdots,N)$ can be determined. If they substituted in (3) we get the Taylor polynomial solution.
The accuracy of this solution can be checked as follows [9]:

If the solution $y(x)$ and its derivatives are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for $x=x_r\in[a,b]$

$E(x_r)=|A(x_r)y''(x_r)+B(x_r)y'(x_r)+C(x_r)y^n(x_r)|\cong 0$

or

$E(x_r)\le 10^{-l}$, (l_r is any positive integer).

If $\max(10^{-l_r})=10^{-l}$ is prescribed, than the truncation limit N is increased until the difference $|E(x_r)|$ at each of the points becomes smaller than 10^{-l}.

4 Numerical example

Consider $xy^{//}+2y^{/}+xy^5=0$, over [0,1] with y(0)=1, $y^{/}(0)=0$.
The exact solution of this problem is $y=(1+\frac{x^2}{3})^{-\frac{1}{2}}$. If we approximate the solution $y(x)$ by the Taylor Matrix method with N=5 we find that

$$\mathbf{A}=\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&\frac{1}{2}&0 \\ 0&0&0&0&0&\frac{1}{6} \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix} 0&2&0&0&0&0 \\ 0&0&2&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&\frac{1}{3}&0 \\ 0&0&0&0&0&\frac{1}{12} \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&\frac{1}{2}&0&0&0 \\ 0&0&0&\frac{1}{6}&0&0 \\ 0&0&0&0&\frac{1}{24}&0 \end{bmatrix}$$

Where $\mathbf{O}=\begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix}$, $\mathbf{Y}=\begin{bmatrix} y^{(0)}(0)\\y^{(1)}(0)\\y^{(2)}(0)\\y^{(3)}(0)\\y^{(4)}(0)\\y^{(5)}(0) \end{bmatrix}$ and $\overline{Y}=\begin{bmatrix} Y_1^{(0)}(0)\\Y_1^{(1)}(0)\\Y_1^{(2)}(0)\\Y_1^{(3)}(0)\\Y_1^{(4)}(0)\\Y_1^{(5)}(0) \end{bmatrix}$.

Then, the matrix equation $U^*Y+C^*\overline{Y}=Q^*$ becomes

$$\begin{bmatrix} 0&2&0&0&0&0 \\ 0&0&3&0&0&0 \\ 0&0&0&2&0&0 \\ 0&0&0&0&\frac{5}{6}&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{bmatrix}\begin{bmatrix} y^{(0)}(0)\\y^{(1)}(0)\\y^{(2)}(0)\\y^{(3)}(0)\\y^{(4)}(0)\\y^{(5)}(0) \end{bmatrix}+\begin{bmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&\frac{1}{2}&0&0&0 \\ 0&0&0&\frac{1}{6}&0&0 \\ 0&0&0&0&\frac{1}{24}&0 \end{bmatrix}\begin{bmatrix} Y^{(0)}(0)\\Y^{(1)}(0)\\Y^{(2)}(0)\\Y^{(3)}(0)\\Y^{(4)}(0)\\Y^{(5)}(0) \end{bmatrix}=\begin{bmatrix} 0\\0\\0\\1\\0 \end{bmatrix}.$$

From the system obtained above, the coefficients $y^{(n)}(0)$ $(n=0,1,2,3,4,5)$ are found as $y^{(2)}(0)=-\frac{1}{3}$, $y^{(3)}(0)=0$, $y^{(4)}(0)=1$, $y^{(5)}(0)=0$. Therefore the solution is

$$y=1-\frac{1}{6}x^2+\frac{1}{24}x^4.$$

Numerical results obtained for $xy^{//}+2y^{/}+xy^5=0$ is given in the following table.

x_i	Taylor MatrixMethod	Exact Solution	Absolute Error
0.0	1	1	0
0.2	0.9934	0.99339	7.3E-07
0.4	0.9744	0.97435	4.5E-05
0.6	0.9454	0.94491	0.00049
0.8	0.9104	0.90784	0.00256
1.0	0.8750	0.86602	0.00897

© 2013 NSP
Natural Sciences Publishing Cor.

5 Conclusion

In this paper we have presented a formula given by (11), (12) and (13) from which one can compute m^{th} order derivative of y^n. The Taylor matrix method avoids the difficulties and massive computational work by determining the analytic approximate solution and provides a reliable technique that requires less work and highly accurate results if compared with the traditional techniques and existing numerical methods. A considerable advantage of the method is also that the Taylor coefficients of the solution are found very easily by using the computer programs.

References

[1] H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, New York, 1962.

[2] S. Chandrasekhar, Introduction to the Study of Stellar Structure, Dover, New York, 1967.

[3] O.U. Richardson, The emission of Electricty from Hot Bodies, London, 1921.

[4] R.P. Kanwal, K.C. Liu, A Taylor expansion approach for solving integral equation, Int. J. Math. Educ. Sci. Technol. **20** (3) (1989) 411-414.

[5] S. Nas, S. Yalcinbas, M. Sezer, A Taylor Polynomial approach for solving high-order linear Fredholm integro-differential equations, Int. J. Math. Educ. Sci. Technol. **31** (2) (2000) 213-225.

[6] M. Sezer, A method for approximate solution of the second order linear differential equations in terms of the Taylor polynomials, Int. J. Math. Educ. Sci. Technol. **27** (6) (1996) 821-834.

[7] M Sezer, the Taylor polynomials solution of Volterra Integral equations, Int. J. Math. Educ. Sci. Technol. **25** (5) (1994) 625-633.

[8] M. Gulsu, M. Sezer, The approximate solution of higher-order linear difference equation with variable coefficients in terms of the Taylor polynomials, App. Math. Comp. **168** (2005) 76-88.

[9] M. Gulsu, M. Sezer, On the solution of the Riccati equation by the Taylor matrix method, App. Math. Comp., in pres, doi:10.1016/j.amc.2005.09.030.

[10] C.Guler, A new numerical algorithm for the Abel equation of the second kind, International journal of Computer Mathematics, Vol. **84**, No. 1, January 2007, 109-119.

[11] N.Guzel, M.Bayram and C.Guler, On the Numerical Solution of Lane-Emden Differential Equations with Index-M, placeFar East Journal of Applied Mathematics, Volume **29**, Issue 2, Pages 185 - 356, 2007.

[12] A. M. Wazwaz, A New Algorithm for Solving Differential Equations of Lane-Emden Type, Appl. Math. Comput. **118** (2-3) (2001), 287-310.

[13] D. Zwillinger, Handbook of Differential Equations, 3rd ed., Academic Press, Boston, MA, 1997.



Coskun Guler is Assistant Professor of Mathematics at Yildiz Technical University. His main research interest are Numerical Methods, Linear Programming and Optimization Theory.



Ibrahim Emiroglu is Associate Professor of Mathematics at Yildiz Technical University. His research interests are in the areas of Image Processing, Programming Languages, Applied Mathematics and Numerical Analysis.



Fatih Tasci is Professor of Mathematics at Yildiz Technical University. His research interest are in the areas of Applied Mathematics and Numerical Methods.



Mustafa Sivri is Professor of Mathematics at Yildiz Technical University. His research interest are Optimization Theory, Linear Programming and Game Theory.

© 2013 NSP
Natural Sciences Publishing Cor.