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Two-parameter nonstandard deformation of 2x2 matrices

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Abstract

We introduce a two-parameter deformation of 2x2 matrices without imposing any condition on the matrices and give the universal R-matrix of the nonstandard quantum group which satisfies the quantum Yang-Baxter relation. Although in the standard two-parameter deformation the quantum determinant is not central, in the nonstandard case it is central. We note that the quantum group thus obtained is related to the quantum supergroup $GL_{p,q}(1|1)$ by a transformation.

I. INTRODUCTION

Recently the matrix groups of all 2x2 nonsingular matrices like $GL(2)$, $GL(1|1)$, *etc.*, were generalized in two ways as the standard deformation [1–3] and h-deformation.[4–7] Both are based on the deformation of the algebra of functions on the groups generated by coordinate functions that commute.

In standard deformation of matrix groups, these commutation relations are determined by a matrix R so that the functions do not commute but satisfy the equation

$$\hat{R}(T \otimes T) = (T \otimes T)\hat{R},$$

such that, they coincide with the matrix groups for particular values of the deformation parameter. In the h-deformation, this property is the same as the standard deformation. The structure of the matrix groups is important in both deformations since the classical (or super) matrix groups are obtained in some limit of the deformation parameters. In this work we shall construct a two-parameter deformation of 2x2 matrices without imposing any such condition on the matrices just as in Ref. 8 and obtain a two-parameter generalization of their results.

We briefly describe the content of this work. In section II we introduce the group $G_{p,q}$ of the 2x2 matrices by using an R matrix. Section III is devoted to the corresponding Hopf algebra. In Sec. IV we give the universal enveloping algebra of this nonstandard quantum group.

II. $G_{p,q}$-MATRICES

Let

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2x2 matrix with entries belonging to an algebra $\mathcal{A}$. We assume that the quantum group equation (no-grading)

$$\hat{R}(T \otimes T) = (T \otimes T)\hat{R} \tag{1}$$

holds, where

$$\hat{R} = \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & p^{-1} - q & 1 & 0 \\ 0 & qp^{-1} & 0 & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix}. \tag{2}$$

Equation (1) explicitly gives the following relations:

$$ab = -qba, \quad db = qbd,$$

$$ac = -pca, \quad dc = pcd, \tag{3}$$
$$bc = pq^{-1}cb, \quad b^2 = 0 = c^2,$$
$$ad = da + (p^{-1} - q)bc,$$

where p and q are non-zero complex numbers with $pq \pm 1 \neq 0$.

It can be checked that the matrix $R = P\hat{R}$, where P is the usual permutation matrix, satisfies the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{4}$$

and the matrix $\hat{R}$ satisfies the braid group equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{5}$$

We now assume that the matrix elements a and d of T are invertible. Then it is possible to define the inverse of T. To this end, we introduce

$$\Delta_1 = ad - p^{-1}bc, \quad \Delta_2 = da + q^{-1}cb. \tag{6}$$

Then one obtains

$$T_R^{-1} = \begin{pmatrix} \Delta_1^{-1}d & -q\Delta_1^{-1}b \\ p\Delta_2^{-1}c & \Delta_2^{-1}a \end{pmatrix}, \tag{7a}$$

as the right inverse of T. After some calculations we get

$$\Delta_1 d = d\Delta_1, \qquad \Delta_2 a = a\Delta_2,$$
$$\Delta_k b = -q^2 b\Delta_k, \qquad \Delta_k c = -p^2 c\Delta_k, \quad k = 1, 2.$$

Using these relations we obtain

$$T_L^{-1} = \begin{pmatrix} d\Delta_1^{-1} & q^{-1}b\Delta_2^{-1} \\ -p^{-1}c\Delta_1^{-1} & a\Delta_2^{-1} \end{pmatrix} = T_R^{-1}. \tag{7b}$$

Thus the proper left and right inverses of T are equal.

It is easily verified that $a^2\Delta_2^{-1}$ for all values of p and q, commutes with a, d, and anticommutes with b, c. Furthermore $a^2\Delta_2^{-1}$ is invertible. Therefore we obtain

$$S(T) = T^{-1} = \begin{pmatrix} d^{-1} & -a^{-1}ba^{-1} \\ -d^{-1}cd^{-1} & a^{-1} \end{pmatrix} \begin{pmatrix} d^2\Delta_1^{-1} & 0 \\ 0 & a^2\Delta_2^{-1} \end{pmatrix}. \tag{8}$$

We now consider the element

$$D(T) = ad^{-1} - bd^{-1}cd^{-1} = a^2\Delta_2^{-1}. \tag{9}$$

$D(T)$ cannot be regarded as a quantum determinant since it anticommutes with b and c. However, we may regard the element

$$\mathbf{D}(T) = a[D(T) - d^{-1}bd^{-1}c]d^{-1} = [D(T)]^2 \tag{10}$$

as the quantum determinant of T where $D(T)$ is given by (9).

In fact, it is easy to check that the matrix elements of the product matrix TT' satisfy relations (3) for any two commuting quantum matrices T and T' whose elements obey (3). As a consequence of this argument, we have the following relation:

$$D(TT') = D(T)D(T').$$

This result means that $\mathbf{D}(T)$ is central.

This case appears strange from the point of view of quantum group theory.[2] However it becomes clear from the point of view of the corresponding two-parameter quantum supergroup.[9] We know, from the work of Ref. 9, that the quantum superdeterminant of any supermatrix in $\mathrm{GL}_{p,q}(1|1)$ belongs to the centre of the algebra generated by the matrix elements of the supermatrix. In the Appendix, we shall show that this nonstandard quantum group is related to the quantum supergroup $\mathrm{GL}_{p,q}(1|1)$ by a transformation. So we may expect that the quantum superdeterminant in two-parameter nonstandard deformation must again be a central element.

Now let the n-th power of T be

$$T^n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}. \tag{11}$$

Then it is easy to check the following relations:

$$\begin{gathered} A_nB_n = -q^nB_nA_n, \quad D_nB_n = q^nB_nD_n, \\ A_nC_n = -p^nC_nA_n, \quad D_nC_n = p^nC_nD_n, \\ B_n^2 = 0 = C_n^2, \quad q^nB_nC_n = p^nC_nB_n, \end{gathered} \tag{12}$$

and

$$A_nD_n = D_nA_n + (p^{-n} - q^n)C_nB_n. \tag{13}$$

The proof of relation (13) is rather lengthy but straightforward.

Let us finally note the following. If the sum $T + T'$ of two $G_{p,q}$ matrices T and T' is required to be a $G_{p,q}$ matrix then the equation

$$\hat{R}'(T \otimes T') = (T' \otimes T)\hat{R}^{-1} \tag{14}$$

holds, where

$$\hat{R}' = \hat{R}^{-1} - (p - q^{-1})I. \tag{15}$$

Equation (14) explicitly reads

$$\begin{gathered}
a'a = pqaa', \quad dd' = pqd'd, \quad d'a = ad', \\
b'a = -pab', \quad c'a = -qac', \quad bd' = pd'b, \\
bb' = -b'b, \quad cc' = -c'c \quad cd' = qd'c, \\
a'b = -qba' + (p^{-1} - q)b'a, \quad a'c = -pca' + (pq - 1)ac', \\
b'c = pq^{-1}cb' + (q^{-1} - p)ad', \quad bc' = pq^{-1}c'b + (p - q^{-1})d'a, \\
db' = qb'd + (q - p^{-1})bd', \quad dc' = pc'd + (pq - 1)d'c, \\
a'd = da' + (p^{-1} - q)(bc' + b'c).
\end{gathered} \tag{16}$$

Note that the matrix $R' = P\hat{R}'$ again satisfies the quantum Yang-Baxter relation (4), where P is the usual permutation matrix.

III. THE HOPF ALGEBRA STRUCTURE OF $\mathbf{G}_{p,q}$

Let $\mathcal{A}$ be an algebra generated by the elements a, b, c and d satisfying the relations (3). Then $\mathcal{A}$ is the quoitent algebra

$$\mathcal{A} = \mathcal{C}[a, b, c, d]/J,$$

where $\mathcal{C}[a, b, c, d]$ is the free non-commutative algebra generated by a, b, c and d and J is the ideal in $\mathcal{C}[a, b, c, d]$ generated by the relations (3).

The usual co-product on the algebra $\mathcal{A}$ is defined by

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$$

such that

$$\Delta(t^i{}_j) = t^i{}_k \otimes t^k{}_j, \quad T = (t^i_j) \tag{17}$$

(sum over repeated indices) and the counit

$$\varepsilon : \mathcal{A} \longrightarrow \mathcal{C}$$

such that

$$\varepsilon(t^i{}_j) = \delta^i{}_j. \tag{18}$$

The algebra $\mathcal{A}$ is now the matrix bialgebra generated by 1 and $T = (t^i_j)$, and it is a Hopf algebra with the antipode $S(T)$ which is given by (8). To give a proof of this, one has to verify the following:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta,$$

$$(\mathrm{id}\otimes\varepsilon)\circ\Delta=(\varepsilon\otimes\mathrm{id})\circ\Delta, \tag{19}$$
$$m\circ[(\mathrm{id}\otimes S)\circ\Delta]=m\circ[(S\otimes\mathrm{id})\circ\Delta],$$

where m denotes the multiplication mapping

$$m(a\otimes b)=ab$$

for any $a,b\in\mathcal{A}$. The proof follows directly.

IV. UNIVERSAL ENVELOPING ALGEBRA OF $\mathbf{G}_{p,q}$

In this section we shall construct the quantum enveloping algebra in analogy with the FRT approach.[2]

We consider the matrices $L^{\pm}$ with the generators $U_{\pm}$, $V_{\pm}$ and $X_{\pm}$,

$$L^{+}=\begin{pmatrix}U_{+} & \lambda X_{+}\\ 0 & V_{+}\end{pmatrix},\quad L^{-}=\begin{pmatrix}U_{-} & 0\\ -\lambda X_{-} & V_{-}\end{pmatrix}, \tag{20}$$

where $\lambda=q-p^{-1}$. The matrices $L^{\pm}$ satisfy the following relations:

$$\hat{R}L_{1}^{\pm}L_{2}^{\pm}=L_{2}^{\pm}L_{1}^{\pm}\hat{R}, \tag{21}$$

$$\hat{R}L_{1}^{+}L_{2}^{-}=L_{2}^{-}L_{1}^{+}\hat{R}, \tag{22}$$

where $L_1=L\otimes I$ and $L_2=I\otimes L$. These relations give

$$[U_{+},U_{-}]=[V_{+},V_{-}]=[U_{\pm},V_{\pm}]=0,$$

$$\begin{aligned}U_{+}X_{\pm}&=-q^{\mp1}X_{\pm}U_{+},\quad V_{+}X_{\pm}=q^{\mp1}X_{\pm}V_{+},\\ U_{-}X_{\pm}&=-p^{\pm1}X_{\pm}U_{-},\quad V_{-}X_{\pm}=p^{\pm1}X_{\pm}V_{-},\\ X_{+}X_{-}-qp^{-1}X_{-}X_{+}&=\frac{U_{+}V_{-}-V_{+}U_{-}}{q-p^{-1}},\quad X_{\pm}^{2}=0.\end{aligned} \tag{23}$$

The coproduct of the generators is given by

$$\Delta(L^{\pm})=L^{\pm}\dot{\otimes}L^{\pm} \tag{24}$$

where $\dot{\otimes}$ denotes tensor product and matrix multiplication. Explicity, the action of the coproduct Δ on the generators is

$$\Delta(U_{\pm})=U_{\pm}\otimes U_{\pm},$$

$$\Delta(V_{\pm})=V_{\pm}\otimes V_{\pm},$$

$$\begin{aligned}\Delta(X_{+})&=X_{+}\otimes U_{+}+V_{+}\otimes X_{+},\\ \Delta(X_{-})&=X_{-}\otimes V_{-}+U_{-}\otimes X_{-}.\end{aligned} \tag{25}$$

The counit is given by

$$\varepsilon(L^{\pm}) = I. \tag{26}$$

Explicitly,

$$\begin{aligned} \varepsilon(U_{\pm}) = \varepsilon(V_{\pm}) = 1, \\ \varepsilon(X_{\pm}) = 0. \end{aligned} \tag{27}$$

The coinverse is given by

$$\begin{aligned} S(U_{\pm}) = U_{\pm}^{-1}, \quad S(V_{\pm}) = V_{\pm}^{-1}, \\ S(X_{+}) = -U_{+}^{-1} X_{+} V_{+}^{-1}, \\ S(X_{-}) = V_{-}^{-1} X_{-} U_{-}^{-1}. \end{aligned} \tag{28}$$

Therefore one can easily verify that the algebra $\mathcal{U}_{p,q}(U_{\pm}, V_{\pm}, X_{\pm})$ is a Hopf algebra generated by 1, $U_{\pm}$, $V_{\pm}$, $X_{\pm}$ satisfying the relations (23).

The coproduct of $U_{\pm}$ and $V_{\pm}$ together with the fact that they commute implies that they can be written as exponentials of commuting operators,

$$\begin{aligned} U_{+} = q^{-\frac{H}{2}} p^{\frac{N}{2}}, \quad U_{-} = p^{\frac{H}{2}} q^{-\frac{N}{2}}, \\ V_{+} = q^{-\frac{H}{2}} p^{-\frac{N}{2}}, \quad V_{-} = p^{\frac{H}{2}} q^{\frac{N}{2}}, \\ [H, N] = 0. \end{aligned} \tag{29}$$

The commutation relations of $U_{\pm}$ and $V_{\pm}$ with $X_{\pm}$ in terms of new generators give the following:

$$\begin{aligned} [H, X_{\pm}] = \pm 2 X_{\pm}, \quad [N, X_{\pm}] = 0, \\ X_{+} X_{-} - q p^{-1} X_{-} X_{+} = \left(\frac{p}{q} \right)^{\frac{H+1}{2}} [N]_{pq} \end{aligned} \tag{30}$$

where

$$[N]_{pq} = \frac{(pq)^{\frac{N}{2}} - (pq)^{\frac{-N}{2}}}{(pq)^{\frac{1}{2}} - (pq)^{\frac{-1}{2}}}. \tag{31}$$

Moreover, the coproduct is now

$$\begin{aligned} \Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H, \\ \Delta(N) = N \otimes \mathbf{1} + \mathbf{1} \otimes N, \\ \Delta(X_{+}) = X_{+} \otimes q^{-\frac{H}{2}} p^{\frac{N}{2}} + q^{-\frac{H}{2}} p^{-\frac{N}{2}} \otimes X_{+}, \\ \Delta(X_{-}) = X_{-} \otimes p^{\frac{H}{2}} q^{\frac{N}{2}} + p^{\frac{H}{2}} q^{-\frac{N}{2}} \otimes X_{-}. \end{aligned} \tag{32}$$

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V. APPENDIX

A. Nonstandard quantum planes

In this section, we shall consider quantum planes which are similar to quantum superplanes introduced by Manin.[3]

(1) Quantum plane A_p: This plane, or, rather the polynomial function ring on it is generated by coordinates x and θ with the commutation rules

$$x\theta = -p\theta x, \quad \theta^2 = 0, \tag{A1}$$

where p is a complex number. The coordinates anticommute for $p = 1$ and commute for $p = -1$.

(2) Quantum plane A_q^*: This plane is generated by coordinates φ and y with commutation rules

$$\varphi^2 = 0, \quad \varphi y = q^{-1} y\varphi \tag{A2}$$

where q is a complex number. The quantum plane A_q^* is dual to the quantum plane A_p.

Note that the relations (A1) and (A2) are equivalent to the relations

$$\hat{R}(X \otimes X) = -q(X \otimes X), \quad \hat{R}(Y \otimes Y) = p^{-1}(Y \otimes Y).$$

B. Nonstandard quantum deformation of 2x2 matrices with nonstandard quantum planes

Let G be a matrix Lie group of rank 2 and T be any element of G, i.e.,

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries belonging to an algebra $\mathcal{A}$.

We consider linear transformations T with the following properties:

$$T : A_p \longrightarrow A_p, \quad T : A_q^* \longrightarrow A_q^*. \tag{A3}$$

The action of T on points of A_p and A_q^* is

$$\begin{pmatrix} \overline{x} \\ \overline{\theta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}, \quad \begin{pmatrix} \overline{\varphi} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi \\ y \end{pmatrix}. \tag{A4}$$

We assume that the matrix elements of T commute with the coordinates of A_p and A_q^*. As a consequence of the linear transformations in (A3) the vectors $\begin{pmatrix} \overline{x} \\ \overline{\theta} \end{pmatrix}$ and $\begin{pmatrix} \overline{\varphi} \\ \overline{y} \end{pmatrix}$ should belong to A_p and A_q^*, respectively. This imposes (p,q)-commutation relations among the entries of T in (3).

Note that it can be checked that the maps

$$\delta : A_p \longrightarrow G \otimes A_p, \quad \delta^* : A_q^* \longrightarrow G \otimes A_q^* \tag{A5}$$

such that

$$\delta(X) = T \otimes X, \quad \text{i.e.} \quad delta(x_i) = t_i^j \otimes x_j, \quad X = \begin{pmatrix} x \\ \theta \end{pmatrix}$$

$$\delta(Y) = T \otimes Y, \quad \text{i.e.} \quad \delta(y_i) = t_i^j \otimes y_j, \quad Y = \begin{pmatrix} \varphi \\ y \end{pmatrix} \tag{A6}$$

define the co-action of the quantum group $G_{p,q}$ on the nonstandard quantum planes A_p and A_q^*, respectively.

Finally, one can show that the matrix quantum group (3) is isomorphic to the quantum supergroup $GL_{p,q}(1|1)$. Indeed, if we define the transformation

$$T' = TD \tag{A7}$$

where T is a matrix whose the matrix elements satisfy (3) and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}, \quad D^2 = I \tag{A8}$$

and we assume that g commutes with a and d, and anticommutes with b and c, then $T' \in GL_{p,q}(1|1)$ as discussed in Ref. 9. In this case $\Delta(T') = T' \dot{\otimes} T'$, *etc.*, are unchanged. One easily sees that when $p = q$, these relations go back to those of Ref. 8.

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