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# Second regularized trace of a differential operator of Sturm–Liouville type

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We consider a Sturm–Liouville-type differential operator defined on a separable Hilbert space with bounded operator coefficient and periodic boundary conditions given on a finite interval. We obtain an explicit formula which we call the second regularized trace formula of the mentioned operator.

**KEYWORDS**

regularized trace, resolvent, spectrum, trace-class operator

**MSC CLASSIFICATION**

47A10, 47B38, 34L05

## 1 | INTRODUCTION

Consider a number $\lambda \in \mathbb{C}$ and the functions $p, q$, and $w$ defined from the interval $[a, b]$ to $\mathbb{R}$. A Sturm–Liouville boundary value problem is a second-order linear ordinary differential equation in the form

$$-(py')' + qy = \lambda wy.$$

Here, the interval $[a, b]$ can be bounded or unbounded. The function $y = y(t)$ must lie in $C^2[a, b]$ (space of functions with second-order derivatives on $[a, b]$) and satisfy the following boundary value conditions:

$$\begin{cases} a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0 \end{cases} ; \ (a_1, a_2), (b_1, b_2) \in \mathbb{R}^2 - \{(0, 0)\}.$$

The Sturm–Liouville problem consists of determining the values $\lambda$ that give non-trivial solutions. Such $\lambda$s are called the eigenvalues of the system; corresponding functions $y$ are called eigenfunctions of the system. Sturm–Liouville problem are divided into two cases. If $[a, b]$ is finite and the coefficient functions are continuous, the problem is called a regular Sturm–Liouville problem. Here, we consider the regular Sturm–Liouville problem. Otherwise, it is called singular. The spectral properties of the Sturm–Liouville operators are found by a connection between the asymptotic behavior of solutions of a differential equation of the above form and the spectral properties of the corresponding differential operator. Since all interesting events occur in the special case $p(t) = w(t) = 1$, this case is usually studied in the literature. If $a_2 = b_2 = 0$, the boundary conditions are called Dirichlet boundary conditions. In this case, $y(a) = y(b) = 0$. Boundary conditions where $a_1 = b_1 = 0$ are called Neumann boundary conditions.

The spectral properties of Sturm–Liouville operators have been an important area of study for many years, especially due to their numerous applications in physics (see Amrein at al. [1], Berezin and Shubin [2], Levitan and Sargsyan [3], and Naimark [4]). For the heat equation, Dirichlet conditions correspond to maintaining a constant temperature at the ends of the rod. Neumann boundary conditions correspond to no heat flow through the ends or insulation conditions as there will be no temperature gradient at these points. More general boundary conditions allow for partially isolated boundaries.

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A frequently encountered type of boundary condition is the periodic boundary condition. Let us consider the heated rod bent to form a circle. Then, the two endpoints are physically the same. Therefore, we would expect the temperature and temperature gradient to behave the same at these points. In this case, we write $y(a) = y(b) = 0$ and $y'(a) = y'(b) = 0$. Boundary value problems using these conditions should be handled in a different way. Note that any second-order linear operator can be converted to the Sturm–Liouville operator form.

Now, let us choose the interval $[a, b]$ as $[0, \pi]$. Let us consider the Hilbert space of the form $\mathcal{H}_1 = L_2([0, \pi]; \mathcal{H})$, where $\mathcal{H}$ is a separable infinitely dimensional Hilbert space. This space is the space of all measurable functions $f$ satisfying the condition

$$\int_0^\pi \|f(t)\|_{\mathcal{H}}^2 dt < \infty.$$

Also, in this space, the inner product is defined by

$$\langle f, g\rangle = \int_0^\pi \langle f(x), g(x)\rangle_{\mathcal{H}} dx \ \ (f, g \in \mathcal{H}_1),$$

where $f, g \in \mathcal{H}_1$. Consider the Sturm–Liouville operator $D$ in $\mathcal{H}_1$ with the periodic boundary conditions:

$$D = -\frac{d^2}{dt^2} + Q,$$
$$y(0) = y(\pi),\ y'(0) = y'(\pi).$$

Assume that the operator valued function $Q(t)$ has the following properties:

- $(q_1)$ $Q(t)$ has a weak second-order derivative in $[0, \pi]$, $Q(0) = Q(\pi)$ and for $t \in [0, \pi]$, $Q^{(i)}(t)$ is a self-adjoint kernel operator on $\mathcal{H}$(i=0, 1, 2).
- $(q_2)$ $\|Q\|_{\mathcal{H}_1} < 2$.
- $(q_3)$ $\mathcal{H}_1$ has an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ such that $\sum_{n=1}^\infty \|Q\varphi_n\|_{\mathcal{H}_1} < \infty$.
- $(q_4)$ $\left\|Q^{(i)}(t)\right\|_{\sigma_1(\mathcal{H})}$ $(i = 0, 1, 2)$ is a bounded measurable function on $[0, \pi]$.

Note that these properties are minimal and necessary for the techniques we use throughout the article. Now, let $D_0$ denote the operator $D$ with $Q(t) = 0$. The operator $D_0$, like the operator $D$, is a self-adjoint operator on $\mathcal{H}_1$, and its spectrum is the set $\sigma(D_0) = \{(2m)^2\}_{m=0}^\infty$. Each $\lambda \in \sigma(D_0)$ is an eigenvalue of infinite multiplicity for $D_0$ and the corresponding orthonormal eigenfunctions to these eigenvalues are of the form

$$\frac{1}{\sqrt{\pi}} \cdot \varphi_n, \sqrt{\frac{2}{\pi}} \cos 2t \cdot \varphi_n, \sqrt{\frac{2}{\pi}} \sin 2t \cdot \varphi_n, \sqrt{\frac{2}{\pi}} \cos 4t \cdot \varphi_n, \sqrt{\frac{2}{\pi}} \sin 4t \cdot \varphi_n \cdots.$$

Let us give some basic concepts related to our problem: Recall that the operator that takes the bounded set to the compact set is called the compact operator or the completely continuous operator. Let $\sigma_\infty(\mathcal{H})$ denote the set of compact operators on $\mathcal{H}$. If $T \in \sigma_\infty(\mathcal{H})$, $T^*T$ is a non-negative self-adjoint operator and $(T^*T)^{1/2} \in \sigma_\infty(\mathcal{H})$, with $T^*$ being the adjoint of the operator $T$. Let $s_1 \geq s_2 \geq \cdots \geq s_k$ $(0 \leq k \leq \infty)$ be the nonzero eigenvalues of the operator $(T^*T)^{1/2}$. Here, each eigenvalue is given as its multiplicity number. The numbers $s_1, s_2, \cdots, s_k$ are called $s$-numbers of the operator $T$ and are denoted by $s_k(T)$ $(k = 1, 2, \cdots)$. When $k < \infty$, $s_j = 0$ is assumed for $j = k + 1, j = k + 2, \cdots$. We denote by $\sigma_p(\mathcal{H})$ the set of all compact operators $T$ defined on $\mathcal{H}$ satisfying the condition $\sum_{k=1}^\infty s_k^p(A) < \infty$ where $s_k(T)$ $(k = 1, 2, \cdots)$ are the $s$-numbers of $T$. If $T \in \sigma_1(\mathcal{H})$, it is called a trace-class or kernel operator. We denote trace of the operator $T$ by $trT$ and define as

$$trT = \sum_{k=1}^{\nu(T)} \lambda_k(T),$$

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where $\nu(T)$ is the sum of algebraic multiplicities of nonzero eigenvalues $\lambda_k(T)$ of the operator $T$. Our main references providing detailed information on the subject are given by Gohberg and Krein [5], Lusternik and Sobolev [6], and Gorbachuk and Gorbachuk [7].

The regularized trace theory of differential operators began in the mid-20th century and took its core from the work of Gelfand and Levitan [8] in which the Sturm–Liouville operator with scalar coefficient was studied. Various regular trace formulas of Sturm–Liouville operators were obtained in the studies of Dikiy [9], Gelfand [10], Halberg and Kramer [11], Levitan [12], Gorbachuk [13], and Fulton and Prues [14].

Regularized trace formulas for the parameter-dependent Sturm–Liouville operator using the Levitan method were developed by Albayrak et al. [15] and Guliyev [16].

The regularized trace formulas of the Schrödinger operator are given in Lax [17].

Regularized trace formulas for various differential operators are presented in the works by Buslaev and Faddeev [18], Gasymov and Levitan [19], Abdukadyrov [20], Sadovnichiĭ and Dubrovskiĭ [21], Fazullin and Murtazin [22], Khasanov and Yakhshimuratov [23], Yang [24], Kanguzhin and Tokmagambetov [25], and Polyakov [26].

Trace formulas for differential operators with operator coefficients are included in the works of Khalilova [27], Adıgüzelov [28], Adıgüzelov et al. [29], Gül [30–33], Bayramov et al. [34], Badalova [35], Karayel and Sezer [36], and Gül and Ceyhan [37]. The second regular trace formula of the Sturm–Liouville operator with inverse periodic boundary conditions and scalar coefficient is given in the study Akgun et al. [38].

The calculation of the second regularized trace formula of the Sturm–Liouville operator with operator coefficient introduced in Adıgüzelov et al. [29] where the second regularized trace formula of the Sturm–Liouville operator with the bounded operator coefficient given by Dirichlet boundary conditions is given. In Gül and Ceyhan [37], second regularized trace formulas were found for a fourth-order differential operator with a bounded operator coefficient given with symmetric boundary conditions, and in Gül [32], second regularized trace formula are found for a high-order differential operator with unbounded operator coefficient given with symmetric boundary conditions. In addition, in Gül and Gill [39, 40], the first regularized trace formulas for differential operators with bounded and unbounded operator coefficients, respectively, are calculated on a separable Banach space. In these articles, the concept of semi-inner product was used in addition to the known methods.

The first regularized trace formula of the Sturm–Liouville operator with a bounded operator coefficient given periodic boundary conditions defined in a separable Hilbert space is recently obtained in Gül [33]. Here, we calculate the second regularized trace formula of the Sturm–Liouville operator with a bounded operator coefficient given by the periodic boundary conditions defined in a separable Hilbert space.

## 2 | MODELING OF THE REGULARIZED TRACE

Let

$$\begin{aligned} \psi_{mn}^{(1)} &= \delta_m \cos 2mt \cdot \varphi_n, \qquad m = 0, 1, 2, \cdots; n = 1, 2, 3, \cdots, \\ \psi_{mn}^{(2)} &= \delta_m \sin 2mt \cdot \varphi_n, \qquad m = 1, 2, 3, \cdots; n = 1, 2, 3, \cdots, \end{aligned} \tag{2.1}$$

where

$$\delta_m = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } m = 0, \\ \sqrt{\frac{2}{\pi}} & \text{if } m = 1, 2, \cdots. \end{cases}$$

We know that the operators $QR_\lambda^0$ and $R_\lambda - R_\lambda^0$ are of the trace class (see Gül [33]). Here, $R_\lambda^0 = (D_0 - \lambda I)^{-1}$, $R_\lambda = (D - \lambda I)^{-1}$ are the resolvents of the operators $D_0$ and $D$, respectively. For simplicity, let $\mu_m = (2m)^2$ $(m = 0, 1, 2, \cdots)$. Now, we have the following:

**Proposition 2.1.** *The spectrum of $D$ is a subset of the union of the pairwise disjoint intervals $F_m = [\mu_m - \|Q\|_{\mathcal{H}_1}, \mu_m + \|Q\|_{\mathcal{H}_1}]$ $(m = 0, 1, 2, \cdots)$ and the following properties hold:*

*(a) Every point of the spectrum that does not belong to the set $\{\mu_m\}_{m=0}^{\infty}$ is a discrete eigenvalue of $D$ with a finite multiplicity.*

*(b) For $m = 0, 1, 2, \cdots$, each $\mu_m$ can be an eigenvalue of $D$ with a finite or infinite multiplicity,*

*(c)*

$$\lim_{n\to\infty} \lambda_{mn} = \mu_m,$$

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*where $\lambda_{mn}$ are the eigenvalues of $D$ in $F_m$ and each eigenvalue has been counted as many as its multiplicity.*

Proof of the proposition is similar to the proof of Theorem 2 in Gül [30].
Since $R_\lambda - R_\lambda^0$ is a trace-class operator, for every $\lambda \in \rho(D)$, we write

$$tr(R_\lambda - R_\lambda^0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda} - \frac{2}{\mu_m - \lambda}\right).$$

Multiplying both sides of the last equality by $\lambda^2/2\pi i$ and integrating over the circle $|\lambda| = b_p = \mu_p + 2p\,(p = 1, 2, \cdots)$, we find

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda^2 tr\left(R_\lambda - R_\lambda^0\right)d\lambda &= \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda^2\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda} - \frac{2}{\mu_m-\lambda}\right)\\ &\quad + \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda^2\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda} - \frac{2}{\mu_m-\lambda}\right).\end{aligned} \tag{2.2}$$

For $m \le p$ and $p \ge 1$, we get

$$\mu_m - 2 \le \lambda_{mn} \le \mu_m + 2 \le \mu_p + 2 \le \mu_p + 2p = b_p,$$

and for $m > p$, we get

$$\lambda_{mn} \ge \mu_m - 2 \ge \mu_{p+1} - 2 > b_p.$$

Thus, we find

$$|\lambda_{mn}| < b_p \;;\; m \le p, p \ge 1, n = 1, 2\cdots, \tag{2.2a}$$

$$\lambda_{mn} > b_p \;;\; m > p, p \ge 1, n = 1, 2\cdots. \tag{2.2b}$$

If the inequalities (2.2a) and (2.2b) are taken into account in Equation (2.2), we obtain

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda^2 tr\left(R_\lambda - R_\lambda^0\right)d\lambda = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(2\mu_m^2 - \lambda_{mn}^2\right).$$

Using the Hilbert identity

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{N}(-1)^j R_\lambda^0\left(QR_\lambda^0\right)^j + (-1)^{N+1}R_\lambda\left(QR_\lambda^0\right)^{N+1},$$

where $N$ is a natural number, we find a modeling of the regularized trace as

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\lambda_{mn}^2 - 2\mu_m^2\right) = M_{pN} + \sum_{j=1}^{N}M_j^p, \tag{2.3}$$

where

$$M_{pN} = \frac{(-1)^N}{2\pi i}\int_{|\lambda|=b_p}\lambda^2 tr\left[R_\lambda\left(QR_\lambda^0\right)^{N+1}\right]d\lambda, \tag{2.4}$$

and

$$M_j^p = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda^2 tr\left[R_\lambda^0\left(QR_\lambda^0\right)^j\right]d\lambda.$$

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Now, since the operator function $QR_\lambda^0$ is analytic in the resolvent set $\rho(D_0)$ with respect to the $\sigma_1(\mathcal{H}_1)$ norm and $\left(QR_\lambda^0\right)' = Q\left(R_\lambda^0\right)^2$ (see Gül [33]), it follows that for every natural number $j$, $\left(QR_\lambda^0\right)^j$ is also analytic in $\rho(D_0)$ with respect to the $\sigma_1(\mathcal{H}_1)$ norm. On the other hand, by using mathematical induction method, we find

$$tr\left\{\left[\left(QR_\lambda^0\right)^j\right]'\right\} = j\,tr\left[\left(QR_\lambda^0\right)'\left(QR_\lambda^0\right)^{j-1}\right],$$

and we write this in the form

$$tr\left\{\left[\left(QR_\lambda^0\right)^j\right]'\right\} = j\,tr\left[R_\lambda^0\left(QR_\lambda^0\right)^j\right],$$

since $\left(QR_\lambda^0\right)' = Q\left(R_\lambda^0\right)^2$. Substituting this relation in expression $M_j^p$, we get

$$M_j^p = \frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_p}\lambda^2 tr\left\{\left[\left(QR_\lambda^0\right)^j\right]'\right\}d\lambda.$$

If we rewrite this formula as

$$M_j^p = \frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_p} tr\left\{[\lambda^2\left(QR_\lambda^0\right)^j]' - 2\lambda\left(QR_\lambda^0\right)^j\right\}d\lambda,$$

we find

$$M_j^p = \frac{(-1)^j}{\pi ij}\int\limits_{|\lambda|=b_p}\lambda tr\left\{\left(QR_\lambda^0\right)^j\right\}d\lambda + \frac{(-1)^{j+1}}{2\pi ij}\int\limits_{|\lambda|=b_p}\left\{tr[\lambda^2\left(QR_\lambda^0\right)^j]\right\}'d\lambda.$$

It is not difficult to show that the second term on the right-hand side of this equality is zero. Therefore, we obtain

$$M_j^p = \frac{(-1)^j}{\pi ij}\int\limits_{|\lambda|=b_p}\lambda\, tr\left(QR_\lambda^0\right)^j d\lambda. \tag{2.5}$$

## 3 | EXPLICIT FORMULA FOR THE REGULARIZED TRACE

Part I: We will estimate the terms $M_1^p$ and $M_2^p$.

**Theorem 3.1.** *If the condition $(q_3)$ is satisfied, $\|Q(t)\|_{\sigma_1(\mathcal{H})}$ is integrable on $[0,\pi]$ and $Q'(0), Q'(\pi) \in \sigma_1(\mathcal{H})$, then the following equality holds:*

$$M_1^p = \frac{4}{\pi}\sum_{m=0}^{p}\mu_m\int\limits_0^\pi trQ(t)\,dt.$$

*Proof.* From (2.5), for $j = 1$, we write

$$M_1^p = -\frac{1}{\pi i}\int\limits_{|\lambda|=b_p}\lambda tr\left(QR_\lambda^0\right)d\lambda.$$

Since the system (2.1) is an orthonormal basis of the space $\mathcal{H}_1$, we have

$$M_1^p = -\frac{1}{\pi i}\int\limits_0^\pi\lambda\left[\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\langle QR_\lambda^0\psi_{mn}^{(1)},\psi_{mn}^{(1)}\right\rangle + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\langle QR_\lambda^0\psi_{mn}^{(2)},\psi_{mn}^{(2)}\right\rangle\right]d\lambda.$$

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By the condition $(q_3)$, we get

$$M_1^p = -\frac{1}{\pi i}\left[\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\int_{|\lambda|=b_p}\lambda(\mu_m-\lambda)^{-1}\left\langle Q\psi_{mn}^{(1)},\psi_{mn}^{(1)}\right\rangle d\lambda+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{|\lambda|=b_p}\lambda(\mu_m-\lambda)^{-1}\left\langle Q\psi_{mn}^{(2)},\psi_{mn}^{(2)}\right\rangle\right]d\lambda.$$

Let us rewrite the last expression as follows:

$$M_1^p = 2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\langle Q\psi_{mn}^{(1)},\psi_{mn}^{(1)}\right\rangle\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\mu_m}d\lambda+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\langle Q\psi_{mn}^{(2)},\psi_{mn}^{(2)}\right\rangle\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\mu_m}d\lambda.$$

By using (2.2a) and (2.2b), we find

$$M_1^p = 2\sum_{m=0}^{p}\sum_{n=1}^{\infty}\mu_m\left\langle Q\psi_{mn}^{(1)},\psi_{mn}^{(1)}\right\rangle+2\sum_{m=1}^{p}\sum_{n=1}^{\infty}\mu_m\left\langle Q\psi_{mn}^{(2)},\psi_{mn}^{(2)}\right\rangle,$$

when $\mu_m=(2m)^2$ $(m=0,1,2,\cdots)$. Thus, we obtain

$$\begin{aligned}
M_1^p &= 2\sum_{m=0}^{p}\sum_{n=1}^{\infty}\mu_m\left[\int_0^{\pi}Q(t)\left\langle\sqrt{2\pi^{-1}}\varphi_n\cos(2mt),\sqrt{2\pi^{-1}}\varphi_n\cos(2mt)\right\rangle_{\mathcal{H}}dt\right.\\
&\quad\left.+\int_0^{\pi}Q(t)\left\langle\sqrt{2\pi^{-1}}\varphi_n\sin(2mt),\sqrt{2\pi^{-1}}\varphi_n\sin(2mt)\right\rangle_{\mathcal{H}}dt\right]\\
&= \frac{4}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\mu_m\left[\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\cos^2(2mt)\,dt\right.\\
&\quad\left.+\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\sin^2(2mt)\,dt\right]\\
&= \frac{4}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\mu_m\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\,dt\\
&= \frac{4}{\pi}\sum_{m=1}^{p}\mu_m\int_0^{\pi}trQ(t)\,dt,
\end{aligned}$$

that is,

$$M_1^p = \frac{4}{\pi}\sum_{m=0}^{p}\mu_m\int_0^{\pi}trQ(t)\,dt. \tag{3.1}$$

□

The following lemma will be used in our further computations.

**Lemma 3.1.** *Let $E=\{(r,m):r,m\in\mathbf{N};\ r-m=i,\ r>p\}$, where $p$ and $i$ are positive integers. Then, we have*

$$\sum_{m,r\in E}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}=\begin{cases}p+\frac{1}{2}+i^2O\left(p^{-1}\right) & \text{if } i\le p,\\ O(p) & \text{if } i>p.\end{cases}$$

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*Proof.* Consider the case where $i \le p$: We observe

$$\begin{aligned}\sum_{\substack{r-m=i\\ m\le p\\ r>p}} \frac{\mu_m}{\mu_r-\mu_m} &= \sum_{\substack{r-m=i\\ m\le p\\ r>p}} \frac{m^2}{r^2-m^2} = \frac{p^2}{(p+i)^2-p^2}\\ &\quad + \frac{(p-1)^2}{(p-1+i)^2-(p-1)^2} + \cdots + \frac{(p-i+1)^2}{(p+1)^2-(p-i+1)^2}\\ &= \sum_{j=0}^{i-1} \frac{(p-j)^2}{(p-j+i)^2-(p-j)^2}\\ &= \frac{p}{2} + \frac{1-2i}{4} + \sum_{j=0}^{i-1} \frac{i}{4(2p-2j+i)}.\end{aligned}$$

Thus, we obtain

$$\sum_{m,r\in E} \frac{\mu_r+\mu_m}{\mu_r-\mu_m} = p + \frac{1}{2} + \frac{i}{2} \sum_{j=p-i+1}^{p} \frac{1}{2j+i} = p + \frac{1}{2} + i^2 O\left(p^{-1}\right).$$

If $i > p$, then we get

$$\sum_{\substack{r-m=i\\ m\le p\\ r>p}} \frac{\mu_m}{\mu_r-\mu_m} = \frac{p(p+1)}{4i} - \frac{p}{4} + \frac{i}{4} \sum_{j=1}^{p} \frac{1}{1+2j},$$

and so

$$\begin{aligned}\sum_{m,r\in E} \frac{\mu_r+\mu_m}{\mu_r-\mu_m} &= \sum_{m,r\in E} \left(1 + \frac{2\mu_m}{\mu_r-\mu_m}\right) = p + 2\sum_{m,r\in E} \frac{\mu_m}{\mu_r-\mu_m}\\ &= p + \frac{1}{2} + \frac{(p+1)(p-i)}{2i} + \frac{i}{2} \sum_{j=1}^{p} \frac{1}{2j+i} = O(p).\end{aligned}$$

□

**Theorem 3.2.** *If the condition* $(q_3)$ *is satisfied,* $\|Q(t)\|_{\sigma_1(\mathcal{H})}$ *is integrable on* $[0,\pi]$, $Q(0) = Q(\pi)$ *and* $Q'(0), Q'(\pi) \in \sigma_1(\mathcal{H})$, *then*

$$\begin{aligned}M_2^p &= \sum_{m=0}^{p}\sum_{n=1}^{\infty} \|Q\psi_{mn}^{(1)}\|^2 + \sum_{m=1}^{p}\sum_{n=1}^{\infty} \|Q\psi_{mn}^{(2)}\|^2 - \frac{2p+1}{\pi}\int_0^{\pi} trQ^2(t)\,dt\\ &\quad + \frac{2p+1}{2\pi^2} tr\left(\int_0^{\pi} Q(t)\,dt\right)^2 + O\left(p^{-1}\right).\end{aligned}$$

*Proof.* By using (2.5), for $j = 2$, we write

$$M_2^p = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda tr\left[\left(QR_\lambda^0\right)^2\right] d\lambda.$$

Therefore, we have

$$M_2^p = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left[\left\langle \left(QR_\lambda^0\right)^2 \psi_{mn}^{(1)}, \psi_{mn}^{(1)}\right\rangle + \left\langle \left(QR_\lambda^0\right)^2 \psi_{mn}^{(2)}, \psi_{mn}^{(2)}\right\rangle\right] d\lambda. \tag{3.2}$$

Furthermore, since

$$QR_\lambda^0 \psi_{mn}^{(i)} = \frac{Q\psi_{mn}^{(i)}}{\mu_m - \lambda} \qquad (i = 1, 2),$$

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and

$$\left(QR_\lambda^0\right)^2\psi_{mn}^{(i)} = (\mu_m-\lambda)^{-1}QR_\lambda^0\left(Q\psi_{mn}^{(i)}\right) \qquad (i=1,2),$$

we find

$$\left(QR_\lambda^0\right)^2\psi_{mn}^{(1)} = (\mu_m-\lambda)^{-1}QR_\lambda^0\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left[\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\right\rangle\psi_{rq}^{(1)}+\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\right\rangle\psi_{rq}^{(2)}\right],$$

and

$$\left(QR_\lambda^0\right)^2\psi_{mn}^{(2)} = (\mu_m-\lambda)^{-1}QR_\lambda^0\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left[\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\right\rangle\psi_{rq}^{(1)}+\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\right\rangle\psi_{rq}^{(2)}\right].$$

Substituting this last two expressions in Equation (3.2), we get

$$\begin{aligned}M_2^p = \frac{1}{2\pi}\int_{|\lambda|=b_p}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}&\left[\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\right\rangle\left\langle Q\psi_{rq}^{(1)},\psi_{mn}^{(1)}\right\rangle+\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\right\rangle\left\langle Q\psi_{rq}^{(2)},\psi_{mn}^{(1)}\right\rangle\right.\\&\left.+\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\right\rangle\left\langle Q\psi_{rq}^{(1)},\psi_{mn}^{(2)}\right\rangle+\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\right\rangle\left\langle Q\psi_{rq}^{(2)},\psi_{mn}^{(2)}\right\rangle\right]\frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)}d\lambda.\end{aligned}$$

Hence, we write

$$\begin{aligned}M_2^p &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left[\left(\left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\right\rangle\right|^2\right)\Delta\right]\\&= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}Q_{mnrq}\Delta+\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}Q_{mnrq}\Delta+\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}Q_{mnrq}\Delta+\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}Q_{mnrq}\Delta,\end{aligned}$$

where

$$Q_{mnrq} = \left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\right\rangle\right|^2,$$

and

$$\Delta = \frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)}d\lambda.$$

Note that

$$\Delta = \begin{cases}1 & \text{if } m\le r,\\ \frac{\mu_m}{\mu_m-\mu_r} & \text{if } m\le p \text{ and } r>p,\\ 0 & \text{if } m,r>p.\end{cases}$$

Therefore,

$$\begin{aligned}M_2^p &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}Q_{mnrq}+2\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}Q_{mnrq}\frac{\mu_m}{\mu_m-\mu_r}\\&= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}Q_{mnrq}-\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}Q_{mnrq}\left(1+\frac{2\mu_m}{\mu_r-\mu_m}\right)\\&= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(1)}\right\|^2+\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(2)}\right\|^2-\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}Q_{mnrq}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}.\end{aligned}$$

Let us denote

$$\alpha_p = \alpha_p(n,q) = \sum_{m=0}^{p}\sum_{r=p+1}^{\infty}Q_{mnrq}\frac{\mu_r+\mu_m}{\mu_r-\mu_m},$$

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with

$$\alpha_p^{(1)} = \alpha_p^{(1)}(n,q) = \sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left(\left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\right\rangle\right|^2\right),$$

$$\alpha_p^{(2)} = \alpha_p^{(2)}(n,q) = \sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left(\left|\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\right\rangle\right|^2+\left|\left\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\right\rangle\right|^2\right).$$

Therefore, we write

$$M_2^p = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(1)}\right\|^2+\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(2)}\right\|^2-\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_p. \tag{3.3}$$

On the other hand, since

$$\left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\right\rangle\right|^2 = \left|\int_0^{\pi}\left\langle Q(t)\delta_m\cos(2mt)\varphi_n,\delta_r\cos(2rt)\varphi_q\right\rangle_{\mathcal{H}}dt\right|^2,$$

and

$$\left|\left\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\right\rangle\right|^2 = \left|\int_0^{\pi}\left\langle Q(t)\delta_m\cos(2mt)\varphi_n,\delta_r\sin(2rt)\varphi_q\right\rangle_{\mathcal{H}}dt\right|^2,$$

for $m=\overline{0,p}$ and $r=\overline{p+1,\infty}$, accordingly, we get

$$\begin{aligned}
\alpha_p^{(1)} &= \frac{1}{2\pi}\sum_{m=0}^{p}\delta_m^2\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos 2(m-r)t\,dt\right|^2+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin 2(m-r)t\,dt\right|^2\right]\\
&\quad+\frac{1}{\pi}\sum_{m=0}^{p}\delta_m^2\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left[Re\left(\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{H}\cos 2(m-r)t\,dt\int_0^{\pi}\left\langle\overline{Q(t)\varphi_n,\varphi_q}\right\rangle_{\mathcal{H}}\cos 2(m+r)t\,dt\right)\right.\\
&\quad\left.+Re\left(\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{H}\sin 2(m-r)t\,dt\int_0^{\pi}\left\langle\overline{Q(t)\varphi_n,\varphi_q}\right\rangle_{\mathcal{H}}\sin 2(m+r)t\,dt\right)\right]\\
&\quad+\frac{1}{2\pi}\sum_{m=0}^{p}\delta_m^2\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos 2(m+r)t\,dt\right|^2+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin 2(m+r)t\,dt\right|^2\right]\\
&= \alpha_{p1}^{(1)}+\alpha_{p2}^{(1)}+\alpha_{p3}^{(1)}.
\end{aligned}$$

For some positive integers $p$ and $i$, we have

$$\begin{aligned}
\alpha_{p1}^{(1)} &= \frac{1}{2\pi^2}\sum_{r=p+1}^{\infty}\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2rt)\,dt\right|^2+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2rt)\,dt\right|^2\right]\\
&\quad+\frac{1}{\pi^2}\sum_{i=1}^{\infty}\sum_{\substack{r-m=i\\ m\le p\\ r>p}}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2rt)\,dt\right|^2\right].
\end{aligned} \tag{3.4}$$

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From Lemma 3.1, we obtain

$$\begin{aligned}
\alpha_{p1}^{(1)} &= \frac{1}{\pi^2}\left(p+\frac{1}{2}\right)\sum_{i=0}^{\infty}\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right. \\
&\quad \left.+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2\right]\left(=\alpha_{p11}^{(1)}\right) \\
&\quad +\frac{1}{\pi^2}\sum_{i=1}^{p} i^2 O\left(p^{-1}\right)\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right. \\
&\quad \left.+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2\right]\left(=\alpha_{p12}^{(1)}\right) \\
&\quad +\frac{1}{\pi^2}\sum_{i=p+1}^{\infty} O(p)\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right. \\
&\quad \left.+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2\right]\left(=\alpha_{p13}^{(1)}\right) \\
&= \alpha_{p11}^{(1)}+\alpha_{p12}^{(1)}+\alpha_{p13}^{(1)}.
\end{aligned}$$

Now, we easily see that

$$\alpha_{p11}^{(1)} = \frac{2p+1}{2\pi}\int_0^{\pi}\left|\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\right|^2 dt - \frac{2p+1}{4\pi^2}\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}} dt\right|^2. \tag{3.5}$$

Furthermore, since $Q(0)=Q(\pi)$, we observe

$$\begin{aligned}
\alpha_{p12}^{(1)} &= \frac{1}{\pi^2}\sum_{i=1}^{p} i^2 O\left(p^{-1}\right)\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2+\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2\right] \\
&= \frac{1}{4\pi^2}\sum_{i=1}^{p} O\left(p^{-1}\right)\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2+\frac{1}{4\pi^2}\sum_{i=1}^{p} O\left(p^{-1}\right)\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2 \\
&< \frac{c_1}{8\pi p}\sum_{i=1}^{p}\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\delta_i\sin(2it)\,dt\right|^2+\frac{c_1}{8\pi p}\sum_{i=1}^{p}\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sqrt{\frac{2}{\pi}}\cos(2it)\,dt\right|^2 \\
&= \frac{c_1}{8\pi p}\int_0^{\pi}\left|\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\right|^2 dt+\frac{c_1}{8\pi p}\sum_{i=0}^{p}\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\delta_i\cos(2it)\,dt\right|^2-\frac{c_1}{8\pi^2 p}\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}} dt\right|^2 \\
&= \frac{c_1}{4\pi p}\int_0^{\pi}\left|\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\right|^2 dt-\frac{c_1}{8\pi^2 p}\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}} dt\right|^2,
\end{aligned}$$

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where $c_1$ is a positive constant. Thus, for a positive constant $c_2$, we find

$$|\alpha_{p12}^{(1)}| < \frac{c_2}{p}\int_0^{\pi}\left|\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\right|^2 dt. \tag{3.6}$$

Since $Q(0) = Q(\pi)$, we also observe

$$\begin{aligned}
|\alpha_{p13}^{(1)}| &= \frac{1}{\pi^2}\left|\sum_{i=p+1}^{\infty} O(p)\left[\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2 + \left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2\right]\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{i=p+1}^{\infty} O(p)\frac{1}{i^2}\left[\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2 + \left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right]\right| \\
&< \frac{c_3 p}{4\pi^2(p+1)^2}\sum_{i=p+1}^{\infty}\left[\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2 + \left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right] \\
&< \frac{c_3}{4\pi^2 p}\sum_{i=1}^{\infty}\left[\left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin(2it)\,dt\right|^2 + \left|\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right],
\end{aligned}$$

where $c_3$ is a positive constant. Thus, for a positive constant $c_4$, we find

$$|\alpha_{p13}^{(1)}| < \frac{c_4}{p}\int_0^{\pi}\left|\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\right|^2 dt. \tag{3.7}$$

Therefore, from (3.5)–(3.7), we have the following:

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p11}^{(1)} &= \frac{2p+1}{2\pi}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}dt\right|^2 - \frac{2p+1}{4\pi^2}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}dt\right|^2 \\
&= \frac{2p+1}{2\pi}\int_0^{\pi}\sum_{n=1}^{\infty}\|Q(t)\varphi_n\|_{\mathcal{H}}^2\,dt - \frac{2p+1}{4\pi^2}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left|\left\langle\left(\int_0^{\pi}Q(t)dt\right)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}dt\right|^2 \\
&= \frac{2p+1}{2\pi}\int_0^{\pi}\sum_{n=1}^{\infty}\|Q(t)\varphi_n\|_{\mathcal{H}}^2\,dt - \frac{2p+1}{4\pi^2}\sum_{n=1}^{\infty}\left\|\left(\int_0^{\pi}Q(t)\,dt\right)\varphi_n\right\|_{\mathcal{H}}^2 \\
&= \frac{2p+1}{2\pi}\int_0^{\pi}trQ^2(t)\,dt - \frac{2p+1}{4\pi^2}tr\left(\int_0^{\pi}Q(t)\,dt\right)^2,
\end{aligned}$$

$$\begin{aligned}
0 < \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}|\alpha_{p12}^{(1)}| &< \frac{c_2}{p}\int_0^{\pi}\sum_{n=1}^{\infty}\left\|Q'(t)\varphi_n\right\|_{\mathcal{H}}^2\,dt \\
&= \frac{c_2}{p}\int_0^{\pi}tr\left[Q'(t)\right]^2dt = O\left(p^{-1}\right),
\end{aligned}$$

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and

$$0 < \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left|\alpha_{p13}^{(1)}\right| < \frac{c_4}{p}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\int_0^{\pi}\left|\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\right|^2 dt = O\left(p^{-1}\right).$$

Since

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1}^{(1)} = \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left(\alpha_{p11}^{(1)} + \alpha_{p12}^{(1)} + \alpha_{p13}^{(1)}\right),$$

we obtain

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1}^{(1)} = \frac{2p+1}{2\pi}\int_0^{\pi} trQ^2(t)\,dt - \frac{2p+1}{4\pi^2}tr\left(\int_0^{\pi}Q(t)\,dt\right)^2 + O\left(p^{-1}\right). \tag{3.8}$$

Now, let us estimate $\alpha_{p2}^{(1)}$ and $\alpha_{p3}^{(1)}$. First, we observe

$$\begin{aligned}\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos 2(m-r)t\,dt\right| &= \left|\frac{1}{2(m-r)}\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin 2(m-r)t\,dt\right| \\ &\le \frac{1}{(m-r)^2}\left[\left|\langle Q'(0)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right| + \left|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right| + \int_0^{\pi}\left|\langle Q''(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right|\,dt\right],\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\left|\int_0^{\pi}\left\langle Q(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\sin 2(m-r)t\,dt\right| &= \left|\frac{1}{2(m-r)}\int_0^{\pi}\left\langle Q'(t)\varphi_n,\varphi_q\right\rangle_{\mathcal{H}}\cos 2(m-r)t\,dt\right| \\ &\le \frac{1}{(m-r)^2}\left[\left|\langle Q'(0)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right| + \left|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right| + \int_0^{\pi}\left|\langle Q''(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right|\,dt\right],\end{aligned} \tag{3.10}$$

since $Q(0) = Q(\pi)$. So, for a constant $a_1$, we get

$$\begin{aligned}|\alpha_{p2}^{(1)}| \le \frac{a_1}{\pi}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{(\mu_r-\mu_m)(m-r)^2(m+r)^2}\Bigg[&\left|\langle Q'(0)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right|^2 + \left|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right|^2 \\ &+ \pi\int_0^{\pi}\left|\langle Q''(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right|^2 dt\Bigg].\end{aligned}$$

On the other hand, we see

$$\begin{aligned}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{(\mu_r-\mu_m)(r-m)^2(r+m)^2} &\le \frac{1}{p^2}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{(\mu_r-\mu_m)}\frac{1}{(r-m)^2} \\ &= \frac{1}{p^2}\sum_{i=1}^{\infty}\sum_{m,r\in E}\frac{\mu_r+\mu_m}{(\mu_r-\mu_m)}\frac{1}{i^2} \\ &= \frac{1}{p^2}\sum_{i=1}^{\infty}\frac{1}{i^2}\sum_{m,r\in E}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}.\end{aligned}$$

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By using Lemma 3.1, we get

$$\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{(\mu_r-\mu_m)(r-m)^2(r+m)^2}<\frac{3}{p}\sum_{i=1}^{\infty}\frac{1}{i^2}<\frac{a_2}{p}. \tag{3.11}$$

Equations (3.9) and (3.10) and this last inequality give

$$\left|\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p2}^{(1)}\right|\le\frac{a_3}{p}\left[tr[Q'(0)]^2+tr[Q'(\pi)]^2+\int_0^{\pi}tr[Q''(t)]^2\,dt\right]\le\frac{a_4}{p}. \tag{3.12}$$

Using the relations (3.9) and (3.10), we get

$$\begin{aligned}\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\cos 2(m+r)t\,dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\sin 2(m+r)t\,dt\right|^2\le\frac{6}{(m+r)^4}\left[\left|\langle Q'(0)\varphi_n,\varphi_q\rangle_H\right|^2\right.\\ \left.+\left|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_H\right|^2+\int_0^{\pi}\left|\langle Q''(t)\varphi_n,\varphi_q\rangle_H\right|^2dt\right].\end{aligned} \tag{3.13}$$

Now, if we consider the inequality

$$\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\frac{1}{(m+r)^4}<\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{(\mu_r-\mu_m)(m-r)^2(m+r)^2},$$

together with (3.13), we get

$$0\le\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p3}^{(1)}\le\frac{a_5}{p}. \tag{3.14}$$

Using (3.12) and (3.14), we find

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_p^{(1)}=\frac{2p+1}{2\pi}\int_0^{\pi}trQ^2(t)\,dt-\frac{2p+1}{4\pi^2}tr\left(\int_0^{\pi}Q(t)\,dt\right)^2+O\left(p^{-1}\right). \tag{3.15}$$

To determine $\alpha_p^{(2)}$, we write

$$\left|\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\rangle\right|^2=\left|\int_0^{\pi}\langle Q(t)\sqrt{2\pi^{-1}}\sin(2mt)\varphi_n,\sqrt{2\pi^{-1}}\cos(2rt)\varphi_q\rangle_{\mathcal{H}}dt\right|^2,$$

and

$$\left|\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\rangle\right|^2=\left|\int_0^{\pi}\langle Q(t)\sqrt{2\pi^{-1}}\sin(2mt)\varphi_n,\sqrt{2\pi^{-1}}\sin(2rt)\varphi_q\rangle_{\mathcal{H}}dt\right|^2,$$

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for $m=\overline{1,p}$ and $r=\overline{p+1,\infty}$. Therefore,

$$\begin{aligned}\alpha_p^{(2)} &= \pi^{-2}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\sin 2(m-r)t dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos 2(m-r)t dt\right|^2\right]\\ &+2\pi^{-2}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left[\mathrm{Re}\left(\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\sin 2(m-r)t dt\int_0^{\pi}\langle\overline{Q(t)\varphi_n,\varphi_q}\rangle_{\mathcal{H}}\sin 2(m+r)t dt\right)\right.\\ &\left.-\mathrm{Re}\left(\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos 2(m-r)t dt\int_0^{\pi}\langle\overline{Q(t)\varphi_n,\varphi_q}\rangle_{\mathcal{H}}\cos 2(m+r)t dt\right)\right]\\ &+\pi^{-2}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\sin 2(m+r)t dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos 2(m+r)t dt\right|^2\right].\end{aligned}$$

Let us write this in the form

$$\alpha_p^{(2)}=\alpha_{p1}^{(2)}+\alpha_{p2}^{(2)}+\alpha_{p3}^{(2)}. \tag{3.16}$$

We write $\alpha_{p1}^{(2)}$ as

$$\alpha_{p1}^{(2)}=\pi^{-2}\sum_{i=1}^{\infty}\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\sin(2it)dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\cos(2it)dt\right|^2\right]\sum_{m,r\in E}\frac{\mu_r+\mu_m}{\mu_r-\mu_m},$$

where $E=\{(r,m):r,m\in\mathbf{N};\ r-m=i,\ r>p\}$. By using Lemma 3.1, we get

$$\begin{aligned}\alpha_{p1}^{(2)} &= \pi^{-2}\sum_{i=1}^{p}\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\sin(2it)dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\cos(2it)dt\right|^2\right]\sum_{m,r\in E}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\\ &+\pi^{-2}\sum_{i=p+1}^{\infty}\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\sin(2it)dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\cos(2it)dt\right|^2\right]\sum_{m,r\in E}\frac{\mu_r+\mu_m}{\mu_r-\mu_m}\\ &=\pi^{-2}\left(p+\frac{1}{2}\right)\sum_{i=1}^{\infty}\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\sin(2it)dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right]\\ &+\pi^{-2}\sum_{i=1}^{p}i^2O\left(p^{-1}\right)\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\sin(2it)dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right]\\ &+\pi^{-2}\sum_{i=p+1}^{\infty}O(p)\left[\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_H\sin(2it)dt\right|^2+\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(2it)\,dt\right|^2\right]\\ &=\alpha_{p11}^{(2)}+\alpha_{p12}^{(2)}+\alpha_{p13}^{(2)}.\end{aligned} \tag{3.17}$$

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We obtain

$$\alpha_{p11}^{(2)} = \frac{2p+1}{2\pi}\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}dt\right|^2 + \frac{2p+1}{4\pi^2}\left|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}dt\right|^2,$$

$$|\alpha_{p21}^{(2)}| < c_5 p^{-1}\int_0^{\pi}\left|\langle Q'(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right|^2 dt,$$

and

$$\left|\alpha_{p13}^{(2)}\right| < c_6 p^{-1}\int_0^{\pi}\left|\langle Q'(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\right|^2 dt.$$

Thus, since

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p11}^{(2)} = \frac{2p+1}{2\pi}\int_0^{\pi} trQ^2(t)dt - \frac{2p+1}{4\pi^2}tr\left(\int_0^{\pi}Q(t)dt\right)^2,$$

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left|\alpha_{p12}^{(2)}\right| = O\left(p^{-1}\right),$$

and

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left|\alpha_{p13}^{(2)}\right| = O\left(p^{-1}\right),$$

we find

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1}^{(2)} &= \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left(\alpha_{p11}^{(2)}+\alpha_{p12}^{(2)}+\alpha_{p13}^{(2)}\right)\\ &= \frac{2p+1}{2\pi}\int_0^{\pi}trQ^2(t)dt - \frac{2p+1}{4\pi^2}tr\left(\int_0^{\pi}Q(t)dt\right)^2 + O\left(p^{-1}\right).\end{aligned} \tag{3.18}$$

Similar to estimations of $\alpha_{p2}^{(1)}$ and $\alpha_{p3}^{(1)}$, we show that

$$\left|\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p2}^{(2)}\right| < c_7 p^{-1} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p3}^{(2)} < c_8 p^{-1}.$$

Therefore, we get

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p}^{(2)} = \frac{2p+1}{2\pi}\int_0^{\pi}trQ^2(t)dt - \frac{2p+1}{4\pi^2}tr\left(\int_0^{\pi}Q(t)dt\right)^2 + O\left(p^{-1}\right). \tag{3.19}$$

Thus, from (3.3), (3.15), and (3.19), we obtain

$$M_2^p = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(1)}\right\|^2 + \sum_{m=1}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(2)}\right\|^2 - \frac{2p+1}{\pi}\int_0^{\pi}trQ^2(t)\,dt + \frac{2p+1}{2\pi^2}tr\left(\int_0^{\pi}Q(t)\,dt\right)^2 + O\left(p^{-1}\right). \tag{3.20}$$

Here, $O\left(p^{-1}\right)$ depending on $p$ is a statement satisfying the inequality $\left|O\left(p^{-1}\right)\right| < c_9 p^{-1}$. □

Part II:We will compute the term $M_3^p$ specially in the following theorem.

**Theorem 3.3.** *With the same hypotheses of Theorem 3.2, we have*

$$\lim_{p\to\infty} M_3^p = 0,$$

*on the circle* $|\lambda| = b_p = \mu_p + 2p$.

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*Proof.* Note that

$$\begin{aligned}M_3^p = -\frac{1}{3\pi i}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda}{(\mu_m-\lambda)(\mu_r-\lambda)((\mu_u-\lambda)}d\lambda\\ *\Big\{\Big\langle\Big\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\Big\rangle\Big[\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(1)}\Big\rangle\psi_{uv}^{(1)}+\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(2)}\Big\rangle\psi_{uv}^{(2)}\Big]\\ +\Big\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\Big\rangle\Big[\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(1)}\Big\rangle\psi_{uv}^{(1)}+\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(2)}\Big\rangle\psi_{uv}^{(2)}\Big],\psi_{mn}^{(1)}\Big\rangle\\ +\Big\langle\Big\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\Big\rangle\Big[\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(1)}\Big\rangle\psi_{uv}^{(1)}+\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(2)}\Big\rangle\psi_{uv}^{(2)}\Big]\\ +\Big\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\Big\rangle\Big[\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(1)}\Big\rangle\psi_{uv}^{(1)}+\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(2)}\Big\rangle\psi_{uv}^{(2)}\Big],\psi_{mn}^{(2)}\Big\rangle\Big\},\end{aligned}$$

where the symbol "$*$" denotes that there are numbers among $\mu_m, \mu_r, \mu_u$ less than or greater than $b_p$. Now, we write

$$M_3^p = \frac{1}{3\pi i}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{u=0}^{\infty} * G(m,r,u)F(m,r,u), \tag{3.21}$$

where

$$\begin{aligned}F(m,r,u) = \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{v=1}^{\infty}\Big[&\Big\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\Big\rangle\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(1)}\Big\rangle\Big\langle Q\psi_{uv}^{(1)},\psi_{mn}^{(1)}\Big\rangle+\Big\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(1)}\Big\rangle\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(2)}\Big\rangle\Big\langle Q\psi_{uv}^{(2)},\psi_{mn}^{(1)}\Big\rangle\\ &+\Big\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\Big\rangle\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(1)}\Big\rangle\Big\langle Q\psi_{uv}^{(1)},\psi_{mn}^{(1)}\Big\rangle+\Big\langle Q\psi_{mn}^{(1)},\psi_{rq}^{(2)}\Big\rangle\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(2)}\Big\rangle\Big\langle Q\psi_{uv}^{(2)},\psi_{mn}^{(1)}\Big\rangle\\ &+\Big\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\Big\rangle\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(1)}\Big\rangle\Big\langle Q\psi_{uv}^{(1)},\psi_{mn}^{(2)}\Big\rangle+\Big\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(1)}\Big\rangle\Big\langle Q\psi_{rq}^{(1)},\psi_{uv}^{(2)}\Big\rangle\Big\langle Q\psi_{uv}^{(2)},\psi_{mn}^{(2)}\Big\rangle\\ &+\Big\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\Big\rangle\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(1)}\Big\rangle\Big\langle Q\psi_{uv}^{(1)},\psi_{mn}^{(2)}\Big\rangle+\Big\langle Q\psi_{mn}^{(2)},\psi_{rq}^{(2)}\Big\rangle\Big\langle Q\psi_{rq}^{(2)},\psi_{uv}^{(2)}\Big\rangle\Big\langle Q\psi_{uv}^{(2)},\psi_{mn}^{(2)}\Big\rangle\Big],\end{aligned}$$

$$G(m,r,u) = \int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)(\lambda-\mu_u)}d\lambda.$$

Since

$$F(m,r,u) = F(m,u,r) = F(r,u,m) = F(r,m,u) = F(u,m,r) = F(u,r,m),$$

by using Cauchy integral formulas, we obtain

$$\begin{aligned}M_3^p = \frac{1}{\pi i}\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{u=p+1}^{\infty}G(m,r,u)F(m,r,u)+\frac{1}{\pi i}\sum_{m=0}^{p}\sum_{u=p+1}^{\infty}G(m,m,u)F(m,m,u)\\ +\frac{1}{\pi i}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\sum_{u=p+1}^{\infty}G(m,r,u)F(m,r,u),\end{aligned}$$

or

$$\begin{aligned}M_3^p = 4\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{u=p+1}^{\infty}\frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)}F(m,r,u)-2\sum_{m=0}^{p}\sum_{u=p+1}^{\infty}\frac{\mu_u}{(\mu_u-\mu_m)^2}F(m,m,u)\\ +2\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\sum_{u=p+1}^{\infty}\frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)}F(m,r,u).\end{aligned} \tag{3.22}$$

Let us call

$$F_2(m,r,u) := F(m,r,u) - F_1(m,r,u), \tag{3.23}$$

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with

$$F_1(m,r,u)=\frac{\delta_m^2\delta_r^2\delta_u^2}{8}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{v=1}^{\infty}\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos 2(m-r)t\,dt \int_0^{\pi}\langle Q(t)\varphi_q,\varphi_v\rangle_{\mathcal{H}}\cos 2(r-u)t\,dt\int_0^{\pi}\langle Q(t)\varphi_v,\varphi_n\rangle_{\mathcal{H}}\cos 2(u-m)t\,dt. \tag{3.24}$$

Now, taking the expressions

$$A_i:=\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\sum_{u=p+1}^{\infty}\frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)}F_i(m,r,u)\ (i=1,2), \tag{3.25}$$

$$B_i:=\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{u=p+1}^{\infty}\frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)}F_i(m,r,u)\ (i=1,2), \tag{3.26}$$

$$C_i:=\sum_{m=0}^{p}\sum_{u=p+1}^{\infty}\frac{\mu_u}{(\mu_u-\mu_m)^2}F_i(m,m,u)\ (i=1,2), \tag{3.27}$$

we rewrite $M_3^p$ as

$$M_3^p=2A_1+4B_1-2C_1+2A_2+4B_2-2C_2. \tag{3.28}$$

From the relations

$$F_1(m,r,u)=F_1(m,u,r)=F_1(r,u,m)=F_1(r,m,u)=F_1(u,m,r)=F_1(u,r,m),$$
$$F_1(m,u,u)=\frac{\delta_u^2}{\delta_m^2}F_1(m,m,u)=F_1(m,m,u)\ (m\geq 1),$$

we find

$$A_1=2\sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>u}}^{\infty}\sum_{u=p+1}^{\infty}\frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)}F_1(m,r,u)+\sum_{m=0}^{\infty}\sum_{\substack{u=p+1\\ r=u}}^{\infty}\frac{\mu_m}{(\mu_m-\mu_u)^2}F_1(m,m,u). \tag{3.29}$$

In a similar way, we get

$$B_i=-\sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{u=p+1}^{\infty}\frac{\mu_u}{(\mu_u-\mu_m)(\mu_u-\mu_r)}F_i(m,r,u)\ (i=1,2). \tag{3.30}$$

Thus, we write $M_3^p$ as follows:

$$M_3^p=4A'-4B'-2C'+2A_2+4B_2-2C_2. \tag{3.31}$$

Here,

$$A'=\sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>u}}^{\infty}\sum_{u=p+1}^{\infty}\frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)}F_1(m,r,u), \tag{3.32}$$

$$B'=\sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{u=p+1}^{\infty}\frac{\mu_u}{(\mu_u-\mu_m)(\mu_u-\mu_r)}F_1(m,r,u), \tag{3.33}$$

$$C'=\sum_{m=0}^{p}\sum_{u=p+1}^{\infty}\frac{1}{(\mu_u-\mu_m)}F_1(m,m,u). \tag{3.34}$$

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Let

$$E_1 = \{(m,r,u) : m,r,u \in \mathbf{N}; r-m=i; u-m=j; m \le p; r,u > p\},$$

for any natural numbers $p, i$ and $j$ such that $i > j$ and $p \ge j$.

Since

$$\sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>u}}^{\infty} \sum_{u=p+1}^{\infty} \frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)} |F_1(m,r,u)| < \infty,$$

we write

$$\begin{aligned} A' &= \sum_{i=2}^{\infty} \sum_{j=1}^{p} \left[ \left( \sum_{m,r,u \in E_1} \frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)} \right) \gamma_{ij} \right] \\ &\quad + \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>u \\ u-m>p}}^{\infty} \sum_{u=p+1}^{\infty} \frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)} F_1(m,r,u) \\ &= A'^1 + A'^2. \end{aligned} \tag{3.35}$$

Here,

$$\begin{aligned} \gamma_{ij} = \frac{1}{\pi^3} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \sum_{v=1}^{\infty} \int_0^{\pi} \langle Q(t)\varphi_n, \varphi_q \rangle_{\mathcal{H}} \cos 2it\, dt \\ \int_0^{\pi} \langle Q(t)\varphi_q, \varphi_v \rangle_{\mathcal{H}} \cos 2(i-j)t\, dt \int_0^{\pi} \langle Q(t)\varphi_v, \varphi_n \rangle_{\mathcal{H}} \cos 2jt\, dt. \end{aligned} \tag{3.36}$$

On the other hand, since

$$\sum_{m,r,u \in E_1} \frac{\mu_m}{(\mu_m-\mu_r)(\mu_m-\mu_u)} = \frac{1}{16i} + \frac{j}{p} O(1),$$

we have

$$A'^1 = \sum_{i=2}^{\infty} \sum_{j=1}^{p} \left[ \left( \frac{1}{16i} + \frac{j}{p} O(1) \right) \gamma_{ij} \right], \tag{3.37}$$

where $O(1)$ is an expression depending on $p, i,$ and $j$ and satisfies the condition $|O(1)| < const$. Now, if we let

$$B'^1 = \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<u \\ u-m \le p}}^{p} \sum_{u=p+1}^{\infty} \frac{\mu_u}{(\mu_u-\mu_m)(\mu_u-\mu_r)} F_1(m,r,u),$$

$$B'^2 = \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<u \\ u-m>p}}^{p} \sum_{u=p+1}^{\infty} \frac{\mu_u}{(\mu_u-\mu_m)(\mu_u-\mu_r)} F_1(m,r,u),$$

then we write (3.33) shortly as

$$B' = B'^1 + B'^2. \tag{3.38}$$

Since

$$\sum_{m=0}^{p} \sum_{r=0}^{\infty} \sum_{u=p+1}^{\infty} \frac{\mu_u}{(\mu_u-\mu_m)(\mu_u-\mu_r)} |F_1(m,r,u)| < \infty,$$

from (3.23) and (3.36), we find

$$B'^1 = \sum_{j=2}^{p} \sum_{i=1}^{p-1} \left( \sum_{m,r,u \in E_2} \frac{\mu_u}{(\mu_u-\mu_m)(\mu_u-\mu_r)} \right) \gamma_{ij}. \tag{3.39}$$

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Here, $E_2$ is the set

$$E_2 = \{(m,r,u) : m,r,u \in \mathbf{N}; r-m=i; u-m=j; m,r \le p; u > p\},$$

for any natural numbers $p, i,$ and $j$ such that $i < j \le p$. Now, it is not difficult to show that

$$\sum_{m,r,u \in E_2} \frac{\mu_u}{(\mu_u-\mu_m)(\mu_u-\mu_r)} = \frac{1}{16j} + \frac{i}{p}O(1). \tag{3.40}$$

Thus, (3.39) takes the form

$$B'^1 = \sum_{j=2}^{p}\sum_{i=1}^{p-1}\left(\frac{1}{16j} + \frac{i}{p}O(1)\right)\gamma_{ij}. \tag{3.41}$$

And thus, from (3.31), (3.34), (3.35), (3.37), (3.38), and (3.41), we find

$$\begin{aligned} M_3^p = {} & 4\sum_{i=2}^{\infty}\sum_{j=1}^{p}\left(\frac{1}{16i} + \frac{j}{p}O(1)\right)\gamma_{ij} - 4\sum_{j=2}^{p}\sum_{i=1}^{p-1}\left(\frac{1}{16j} + \frac{i}{p}O(1)\right)\gamma_{ij} + 4A'^2 - 4B'^2 - 2C' \\ & + 2A_2 + 4B_2 - 2C_2. \end{aligned} \tag{3.42}$$

For the operator function $Q(t)$ having second-order continuous derivative with respect to $\sigma_1(\mathcal{H})$ norm on $[0,\pi]$, we have

$$|\gamma_{ij}| \le \frac{const.}{i^2 j^2}.$$

Now, by this inequality, since

$$\left|\sum_{i=2}^{\infty}\sum_{j=1}^{p}\frac{O(1)}{pi^2 j}\right| \le const.p^{-1}\left(\sum_{i=1}^{\infty} i^{-2}\right)\sum_{j=1}^{p} j^{-1} < const.p^{-1}\ln p,$$

we find

$$\sum_{i=2}^{\infty}\sum_{j=1}^{p}\left(\frac{1}{16i} + \frac{j}{p}O(1)\right)\gamma_{ij} = \sum_{i=2}^{\infty}\sum_{j=1}^{p}\frac{\gamma_{ij}}{16i} + o(1), \tag{3.43}$$

where $o(1)$ depends on $p$ and satisfies the condition $\lim_{p\to\infty} o(1) = 0$.

Moreover, since $\gamma_{ij} = \gamma_{ji}$, we get

$$\sum_{j=2}^{p}\sum_{i=1}^{p-1}\left(\frac{1}{16i} + \frac{i}{p}O(1)\right)\gamma_{ij} = \sum_{i=2}^{p}\sum_{j=1}^{p-1}\frac{\gamma_{ij}}{16i} + o(1).$$

Since

$$\left|\sum_{i=2}^{p}\frac{\gamma_{ip}}{i}\right| \le \sum_{i=2}^{p}\frac{1}{i^3 p^2} = o(1),$$

and

$$\left|\sum_{i=p+1}^{\infty}\sum_{j=1}^{p}\frac{\gamma_{ij}}{i}\right| \le \sum_{i=p+1}^{\infty}\sum_{j=1}^{p}\frac{1}{i^3 j^2} = o(1),$$

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we find

$$\sum_{i=2}^{p}\sum_{j=1}^{p-1}\frac{\gamma_{ij}}{16i}=\sum_{i=2}^{\infty}\sum_{j=1}^{p}\frac{\gamma_{ij}}{16i}+o(1). \tag{3.44}$$

In this case, from (3.43) and (3.44), we rewrite (3.42) as

$$M_3^p=4A'^2-4B'^2-2C'+2A_2+4B_2-2C_2+o(1). \tag{3.45}$$

Now, it is not difficult to show that each term on the right-hand side of (3.45) tends to zero as $p\to\infty$. Hence, we conclude

$$\lim_{p\to\infty}M_3^p=0. \tag{3.46}$$

□

Part III:We will first show that $\lim_{p\to\infty}M_j^p=0$ for $j\geq 4$ and $\lim_{p\to\infty}M_{pN}=0$ for $N\geq 4$, and we will find the regularized trace formula for the operator $D$ by using the results in the previous two parts.

In a similar way to the proof of Theorem 3.3, we get

$$\lim_{p\to\infty}M_4^p=\lim_{p\to\infty}M_5^p=0. \tag{3.47}$$

We remark that for a positive constant $c$, the inequalities

$$\|QR_\lambda^0\|_{\sigma_1(\mathcal{H})}<c;\ \|R_\lambda^0\|<cp^{-1};\ \|R_\lambda^0\|<cp^{-1}, \tag{3.48}$$

are satisfied on $|\lambda|=b_p$ (see [31]). Thus, from (2.5), it follows that

$$|M_j^p|\leq c_{10}p^{5-j}\ (c_{10}>0).$$

This means that

$$\lim_{p\to\infty}M_j^p=0\ (j\geq 6). \tag{3.49}$$

Moreover, from Equation (2.4) and the inequalities (3.48), we immediately get

$$|M_{pN}|\leq c_{11}p^{5-N}\ (c_{11}>0).$$

So we also find

$$\lim_{p\to\infty}M_{pN}=0\ (N\geq 6). \tag{3.50}$$

Now, we are ready to express the regularized trace formula for the operator $D$.

**Theorem 3.4.** *If conditions $(q_1)$–$(q_4)$ on $Q(t)$ are fulfilled, then we have the following regularized trace formula for the operator $D$:*

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}^2-2\mu_m^2\right)-\frac{4\mu_m}{\pi}\int_0^\pi trQ(t)\,dt-\frac{1}{\pi^2}tr\left(\int_0^\pi Q(t)\,dt\right)^2\right]=-\frac{1}{2\pi^2}tr\left(\int_0^\pi Q(t)\,dt\right)^2.$$

*Proof.* From Equation (2.3), Theorems 3.1–3.3, (3.47), (3.49), and (3.50), we find

$$\begin{aligned}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\lambda_{mn}^2-2\mu_m^2\right)&=\frac{4}{\pi}\sum_{m=0}^{p}\mu_m\int_0^\pi trQ(t)\,dt\\&+\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(1)}\right\|^2+\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(2)}\right\|^2-\frac{2p+1}{\pi}\int_0^\pi trQ^2(t)\,dt\\&+\frac{2p+1}{2\pi^2}tr\left(\int_0^\pi Q(t)\,dt\right)^2+o(1).\end{aligned} \tag{3.51}$$

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Now, let us calculate the second and third terms on the right-hand side of (3.51):

$$\begin{aligned}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(1)}\right\|^2 &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}\langle Q(t)\delta_m\cos 2mt\varphi_n, Q(t)\delta_m\cos 2mt\varphi_n\rangle_{\mathcal{H}}\,dt\\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}\langle Q^2(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\,dt + \frac{2}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}\cos^2 2mt\langle Q^2(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\,dt\\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}\langle Q^2(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\,dt + \frac{1}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}(1+\cos 4mt)\langle Q^2(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\,dt\\ &= \frac{p+1}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}\langle Q^2(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\,dt + \frac{1}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}\langle Q^2(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\cos 4mt\,dt\\ &= \frac{p+1}{\pi}\int_0^{\pi}trQ^2(t)\,dt + \frac{1}{\pi}\sum_{m=1}^{p}\int_0^{\pi}trQ^2(t)\cos 4mt\,dt,\end{aligned}$$

and similarly,

$$\begin{aligned}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left\|Q\psi_{mn}^{(2)}\right\|^2 &= \frac{2}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}\sin^2 2mt\langle Q^2(t)\varphi_n,\varphi_n\rangle_{\mathcal{H}}\,dt\\ &= \frac{p}{\pi}\int_0^{\pi}trQ^2(t)\,dt - \frac{1}{\pi}\sum_{m=1}^{p}\int_0^{\pi}trQ^2(t)\cos 4mt\,dt.\end{aligned}$$

Thus, Equation (3.51) takes the form

$$\begin{aligned}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\lambda_{mn}^2-2\mu_m^2\right) &= \frac{4}{\pi}\sum_{m=0}^{p}\mu_m\int_0^{\pi}trQ(t)\,dt\\ &\quad+\frac{p}{\pi^2}tr\left(\int_0^{\pi}Q(t)\,dt\right)^2+\frac{1}{2\pi^2}tr\left(\int_0^{\pi}Q(t)\,dt\right)^2+o(1).\end{aligned}$$

Moreover, consider the second term on the right-hand side as

$$\frac{p}{\pi^2}tr\left(\int_0^{\pi}Q(t)dt\right)^2=\frac{1}{\pi^2}\sum_{m=0}^{p}tr\left(\int_0^{\pi}Q(t)dt\right)^2-\frac{1}{\pi^2}tr\left(\int^{\pi}Q(t)dt\right)^2,$$

we get

$$\sum_{m=0}^{p}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}^2-2\mu_m^2\right)-\frac{4}{\pi}\mu_m\int_0^{\pi}trQ(t)dt-\frac{1}{\pi^2}tr\left(\int_0^{\pi}Q(t)dt\right)^2\right]=-\frac{1}{2\pi^2}tr\left(\int_0^{\pi}Q(t)dt\right)^2+o(1).$$

Finally, taking the limit on the both sides of this equation as $p\to\infty$, we obtain the regularized trace formula as desired. □

## AUTHOR CONTRIBUTIONS

**Erdal Gül:** Conceptualization; investigation; methodology; validation; formal analysis; project administration; resources; writing—original draft; writing—review and editing. **Serpil Karayel:** Investigation; formal analysis; writing—original draft; writing—review and editing; methodology; conceptualization; validation; resources.

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## CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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