MAT 5120 - Advanced Algebra - 2020-2021 Spring

Homework Assignment 1

Due April 1st 2021

There are 10 questions each worth 10 points.

(1) Let $G = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$.
(a) Prove that G is a group under addition.
(a) Prove that $G - \{0\}$ is a group under multiplication.

(2) Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$.
(a) Show that for any element $x = sr^i \in D_{2n}$, where $0 \leq i \leq n-1$, we have $rx = xr^{-1}$.
(b) Show that any element $x = sr^i \in D_{2n}$, where $0 \leq i \leq n-1$, has order 2.
(c) Show that if $n = 2k$ is even and $n \geq 4$, then $z = r^k$ is an element of order 2 which commutes with all elements of D_{2n}, and z is the only nonidentity element of D_{2n} which commutes with all elements of D_{2n}.
(d) Show that if n is odd and $n \geq 3$, then the identity is the only element of D_{2n} which commutes with all elements of D_{2n}.

(3) **(a)** Find the order of $(1\ 8)(2\ 11\ 3\ 7\ 10)(4\ 9\ 15)(5\ 17\ 16\ 13)(6\ 12\ 14) \in S_{17}$.
(b) Let $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12) \in S_{12}$. For which positive integers i is σ^i also a 12-cycle?
(c) Let $\tau = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)(13\ 14) \in S_{14}$. Is there a cycle $\sigma \in S_{14}$ such that $\tau = \sigma^k$ for some integer k.
(d) Let $\tau = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15) \in S_{15}$. Is there a cycle $\sigma \in S_{15}$ such that $\tau = \sigma^k$ for some integer k.

(4) **(a)** Find all numbers n such that S_7 contains an element of order n. Explain your answer.
(b) Find the smallest value of n such that S_n contains an element of order 35. Explain your answer.

(5) **(a)** Write out the group tables for S_3, D_6, D_8, and Q_8.

(6) **(a)** Show that S_3 and D_6 are isomorphic.
(b) Show that D_8 and Q_8 are nonisomorphic.

(7) Let $G = GL_2(\mathbb{F}_2)$.
(a) Show that $|G| = 6$.
(b) Write out all elements of G and compute their orders.
(c) Show that G is nonabelian.
(d) Show that $G \cong S_3$.

(8) Let G be a group.

(a) Show that for any $x \in G$, $|x| = |x^{-1}|$.

(b) Let $\phi : G \to G$ be defined as $\phi(x) = x^{-1}$. Show that ϕ is a homomorphism if and only if G is abelian.

(9) Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$, let $k \in \mathbb{Z}$ where $k > 1$, and define $\phi : G \to G$ as $\phi(z) = z^k$. Show that ϕ is a surjective homomorphism but is not an isomorphism.

(10) Let A be a nonempty set, k be a positive integer with $k \leq |A|$, and $B = \{X \subseteq A : |X| = k\}$. The symmetric group S_A acts on B by $\sigma \cdot \{a_1, a_2, \ldots, a_k\} = \{\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)\}$.

(a) Show that this is a group action.

(b) Describe explicitly how the elements (1 2) and (1 2 3) act on the six 2-element subsets of $\{1, 2, 3, 4\}$.