**Definition**: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation (DE).

(1) A DE is an equation involving an unknown function and its derivatives.

Some Examples of Differential Equations:

1. \( \frac{d^2y}{dx^2} + xy \left( \frac{dy}{dx} \right)^2 = 0 \)

2. \( \frac{d^4x}{dt^4} + 5 \frac{d^2x}{dt^2} + 3x = \sin t \) (4th derivative of \( x \) wrt \( t \))

3. \( \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \)

4. \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \)

\( \Rightarrow S^1 = f(x, y) \) standard form

\( \Rightarrow M(x, y)dx + N(x, y)dy = 0 \) differential form.

It is clear that the various variables and derivatives involved in a differential equation can occur in a variety of ways. For example, in the first DE, there exists one independent variable (its order and its power) clearly,
some kind of classification must be made.

(A) DESs can be classified into two groups with respect to the number of independent variable(s).

\[ \text{DESs} \]

\[ \text{Ordinary DESs} \]
\[ f(x, y', y'', \ldots, y^{(n)}) = 0 \]

involves one independent variable

\[ \text{Partial DESs} \]
\[ F(x, y, z, z_x, z_y, z_{xx}, \ldots) = 0 \]

involves two or more independent variables

(A) In this course, we will be concerned only with ordinary DESs.

**Definition**: The order of a DE is the order of the highest derivative appearing in the equation. (Partial)

**Order**: [WolframAlpha](http://www.wolframalpha.com)

**Definition**: A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation (ODE).

**E.g. (For example)**: (1) \[ \frac{d^2y}{dx^2} + xy \left( \frac{dy}{dx} \right)^2 = 0 \] and
(2) \[ \frac{d^4x}{dt^4} + 5 \frac{d^2x}{dt^2} + 3x = 5 \text{int} \] are ODES.

In (1), the variable \( x \) is the single independent variable and \( y \) is a dependent variable.
In (2), the only independent variable is \( t \), whereas \( x \) is dependent.

Definition: A diff. Equation involving partial derivative of one or more dependent variables with respect to more than one independent variable is called a partial diff. equation (PDE).

Eg: (3) \( \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \) and (4) \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \) are PDEs.

In (3), the variables \( s \) and \( t \) are independent variables; and \( v \) is a dependent variable.

In (4), there are three independent variables; \( x, y \) and \( z \); in this equation \( u \) is dependent variable.

A DE is an ordinary diff. equation if the unknown function depends on only one independent variable.

If the unknown function depends on two or more independent variables, the diff. equation is a partial DE. DEs can be classified according to the order of highest derivative appearing in the equation. For this purpose, the following def. must begin:

Definition: The order of the highest derivative involved in a diff. equation is called the order of DE. (Order = most - before)

The order of a DE is the order of the highest derivative appearing in the equation.

Eg: The ordinary DE (1) \( \frac{d^2 y}{dx^2} + xy (\frac{dy}{dx})^2 = 0 \) is of second derivative of \( y \), w.r.t. \( x \).

The second order since the highest derivative involved is a second derivative.

(2) \( \frac{d^4 x}{dt^4} + 5 \frac{d^2 x}{dt^2} + 3x = \sin t \). This DE is of the fourth order. Fourth derivative of \( x \) w.r.t. \( t \).
The Partial DES (3) and (4) are of the first and second orders, respectively.

\[ 3 \text{ DES} \]

The First order DESs High order DESs.

**Definition:** The degree (degree) of a DE is the degree of the highest derivative appearing in the equation.

DEs can be classified into two groups with respect to the degree of the equation.

\[ \text{DES} \]

Linear DES Non-linear DES

**Example:** \((1) \left( \frac{d^2y}{dx^2} \right)^4 + xy \left( \frac{dy}{dx} \right)^2 = 0\) is a non-linear DE since the derivative \( \frac{dy}{dx} \) is of the second degree.

(2), (3), (4) are linear DESs. The dependent variable \( y \) and its various derivatives occur to the first degree only. Each coefficient must be a function of \( x \). No products of \( y \) and/or any of its derivatives are present.

**Definition:** A linear ODE of order \( n \), in the dependent variable \( y \) and the independent variable \( x \), is an equation that is in the form

\[
a_0(x) \frac{d^ny}{dx^n} + a_1(x) \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y = b(x)\]

where \( a_0 \) is not identically zero.
**Example:** The following ODEs are both linear.

\[
\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0, \quad \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} + x^3 \frac{dy}{dx} = xe^x
\]

- In each case, \( y \) is the dependent variable.
- Observe that, \( y \) and its various derivatives occur to the first degree only and that no products of \( y \) and/or any of its derivatives are present.

**Definition:** A nonlinear ODE is an ODE that is not linear.

**Example:** The following ODEs are all nonlinear.

- \( \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \) non-linear b/c the dependent variable \( y \) appears to the second degree in the term 6\( y^2 \).
- \( \frac{d^3y}{dx^3} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0 \) owes its nonlinearity to the presence of the term \( 5\left(\frac{dy}{dx}\right)^3 \), which involves the third power of the first derivative.
- \( \frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \) is nonlinear b/c of the term \( 5y\frac{dy}{dx} \), which involves the product of the dependent variable and its first derivative.

(4) Also ODEs can be classified into two groups with respect to the nature of the coefficients of the dependent variable and its derivatives.

- **DEs with constant coefficients**
  - (Sabit karsoyili)
- **DEs with undetermined coefficients**
  - (Belirsiz degisken)
  - (karsoyili)
Example: Classify each of the following DEs.

\[ \frac{dy}{dx} + x^2 y = xe^x \quad \text{**First order, linear ODE, with undetermined coefficients.} \]

\[ \frac{d^3y}{dx^3} + 4 \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 2y = \sin x \quad \text{**Third order, linear ODE with constant coefficients.} \]

Solutions of a Diff. Equation

**Definition:** A solution to a DE on an interval I is any function \( y = f(x) \) which is defined in I and satisfies the DE.

A solution of a DE in the unknown function \( y \) and the independent variable \( x \) on the interval I is a function \( y(x) \) that satisfies the DE identically for all \( x \) in I.

The graph of \( y = f(x) \) is called as integral curves.

For example: \( y = xe^x \) is a solution of the following DE:

\[ y'' - 2y' + y = 0 \]

Because, for \( x \in \mathbb{R} \), \( y' = xe^x + e^x \) (e.g., \( e^{x+1} \))

\[ y'' = xe^x + 2e^x \]

If these derivatives and the function \( y \) are substituted in DE, equation will be satisfied.

\[ y'' - 2y' + y = xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0 \]
Definition (General Solution): Consider the following $n$th order ODE:

$$f(x, y, y', y'', \ldots, y^{(n)}) = 0 \quad \text{(1)}$$

and the following family of function with $n$ parameters:

$$F(x, y, c_1, c_2, \ldots, c_n) = 0$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

If each function in this family is a solution of the relevant DE $(1)$, then this family is called as the general solution of DE.

Also, it must be noted here that the number of the arbitrary constants in $F$ is equal to the order of DE.

Solutions obtained from integrating the DEs are called general solutions. The general solution of a $n$th order ODE contains $n$ arbitrary constants resulting from integrating $n$ times.

Definition (Particular Solution): Particular solutions are the solutions obtained by assigning specific values to the arbitrary constants in the general solution.

Definition (Singular Solution): Solutions that cannot be expressed by the general solutions are called singular solutions.
Example: Find the general solution of the following DE and specify some particular solutions.

\[ y' = 2x \]

\[ \frac{dy}{dx} = 2x \]

\[ \int dy = \int 2x \, dx + C \]

\[ y = x^2 + C \rightarrow \text{general solution (one-parameter family of solutions)} \]

- \( c = 0 \) \( \Rightarrow \) \( y = x^2 \)
- \( c = 1 \) \( \Rightarrow \) \( y = x^2 + 1 \)
- \( c = 2 \) \( \Rightarrow \) \( y = x^2 + 2 \)

(*) Please note that some DEs have infinitely many solutions (our example), whereas some DEs have no solutions. Example: \( (y')^4 + y^2 = -1 \) - the left side of the DE must be nonnegative for every real function \( y(x) \) and any \( x \), since it is the sum of terms raised to the second and fourth powers, while the right side of the DE is negative. Since no function \( y(x) \) will satisfy this equation, the given DE has no solutions.
Initial-Value and Boundary-Value Problems
(Baslangic-deger ve Sınıf-deger Problemleri)
A DE along with some conditions on the unknown
function and its derivatives, all given at the same
value of the independent variable constitutes an
initial-value problem. The conditions are entitled as
initial conditions. If the conditions are given at
more than one value of the independent variable
the problem is a boundary-value problem and the
conditions are boundary
conditions.

Example: $y'' + 2y = e^x$, $y(0) = 1$, $y'(0) = 2$
(Initial value problem) two conditions are given
at $x = x$

$y'' + 2y = e^x$, $y(0) = 1$, $y(1) = 1$
(boundary value problem) two conditions are given
at $x = 0$ and $x = 1$.

A solution to an initial-value or boundary-value
problem is a function $y(x)$ that both solves
the DE and satisfies all given conditions.

Methods of Solution

There is no general procedure for solving a DE.
Only a few simple equations can be solved
integrating directly. Most equations are solved
by techniques devised for a particular type
of equation. Each of the various types have its
own method of solution.

Generally, to solve an equation, one must be
able to recognize the type and recall the proper
method for solving it.
Example: Obtain the DE that has the following general solution.

\[ y = \sin x - 1 + c \cdot e^{-\sin x} \]

The general solution have one arbitrary constant so it should be derived once.

\[ y' = \cos x - c \cdot \cos x e^{-\sin x} \]

We want the arbitrary constant to be disappeared so if we leave alone \( c \) in the general solution

\[ y = \sin x - 1 + ce^{-\sin x} \implies c = \frac{y - \sin x + 1}{e^{-\sin x}} \]

We substitute this in the first derivative:

\[ y' = \cos x - \left( \frac{y - \sin x + 1}{e^{-\sin x}} \right) \cos x e^{-\sin x} \]

\[ y' = \cos x \left[ 1 - \left( \frac{y - \sin x + 1}{e^{-\sin x}} \right) \right] \]

\[ y' = \cos x \left[ y - y + \sin x - 1 \right] \]

\[ y' = \cos x (\sin x - y) \]
Example: \( y = \frac{c_1}{x} + c_2 \). This equation has 2 constants so it should be derived twice.

\[
\begin{align*}
y &= \frac{c_1}{x} + c_2 \\
\frac{dy}{dx} &= -\frac{c_1}{x^2} \\
\frac{d^2y}{dx^2} &= +\frac{2c_1}{x^3} \\
\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} &= 0
\end{align*}
\]