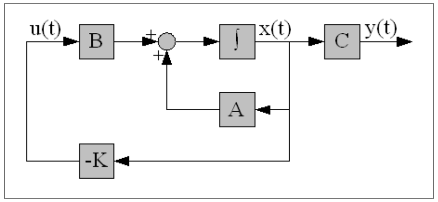
LQR,LQG

1 Symbols

- $x(t) \in \mathbb{R}^n$ State variables
  $u(t) \in \mathbb{R}^m$ System input
  $y(t) \in \mathbb{R}^p$ System output
  $r(t) \in \mathbb{R}^p$ Reference point
  $e(t) \in \mathbb{R}^p$ Error
- $\dot{x}(t) = A\,x(t) + B\,u(t)$
  $y(t) = C\,x(t)$
- $A \in \mathbb{R}^{n \times n}$
  $B \in \mathbb{R}^{n \times m}$
  $C \in \mathbb{R}^{p \times n}$
- The symbol 0 is used for the zero matrix with appropriate dimensions!

2 State Feedback(LQR)

Task



Find the gain matrix K such that the following quality criterion is minimized:

$$J(x(t), u(t)) = \int_0^\infty \left(x^T(t) \cdot Q \cdot x(t) + u^T(t) \cdot R \cdot u(t) \right) \cdot dt$$

Note: the LQR has no reference signal!

Solution

- Find the positive-definit matrix Φ that satisfies following equation:
  $\Phi \cdot B \cdot R^{-1} \cdot B^T \cdot \Phi - \Phi \cdot A - A^T \cdot \Phi - Q = 0$ (Ricatti Equation)
- Compute K:
  $K = R^{-1} \cdot B^T \cdot \Phi$
- The control input is then as follows:
  $u(t) = -K \cdot x(t)$

Tuning

- $Q \in \mathbb{R}^{n \times n}$ tells us how to penalize each of the state variable $x(t)$. In some cases also the square root of Q is used $Q = \bar{C}^T \bar{C}$. In general $\bar{C}$ has nothing to do with the state space matrix C. A good ansatz is to choose a diagonal Matrix for Q, so each element can be interpreted as the factor corresponding to each of the n state variables.
- $R \in \mathbb{R}^{m \times m}$ penalizes the input $u(t)$. Here the ansatz can be chosen as follows: $R = r \cdot I$
- If R is large compared to Q, then we talk about expensive control. The resulting system is rather slow.
- If R is small compared to Q, then we talk about cheap control. The resulting system is fast and has quite good robustness. But be careful not to choose R too small. There is a point where the system loses its robustness and where the noise attenuation is lost.

Conditions

- (A, B) must be controllable
- $(A, \bar{C})$ must be observable ($Q = \bar{C}^T \bar{C}$). This condition is also met if $\bar{C}$ has full rank.

Open-Loop

Open-loop state space matrices:

$$A_{LQR,OL} = A, \quad B_{LQR,OL} = B, \quad C_{LQR,OL} = K, \quad D_{LQR,OL} = 0$$

Loop gain:

$$L_{LQR} = B \cdot (sI - A)^{-1} \cdot K$$

Closed-Loop

Closed-loop state space matrices (There is no input to the system, so B and D don't really exist):

$$A_{LQR,CL} = A - BK, \quad (B_{LQR,CL} = B), \quad C_{LQR,CL} = C, \quad (D_{LQR,CL} = 0)$$

Transfer function (also this does not really exist as there is no input):

$$T_{LQR} = B \cdot (sI - (A - BK))^{-1} \cdot C$$

The resulting system is always asymptotically stable!

Robustness

It is possible to show that following equation holds:

$$\min_{\omega} \sigma_{min}(I + L_{LQR}(i\omega)) = 1$$

This means that the minimum distance to the critical point is always at least 1. Hence, the LQR controller guarantees a very high robustness.

3 LQR-I

Aim

It is possible to include some integral action in the system. This is done so that if we want to achieve reference tracking the steady state error goes towards 0. Therefore generally the output of the system is integrated and fed back through the matrix K_I. (An alternative is to use feedforward action, see chapter 6).

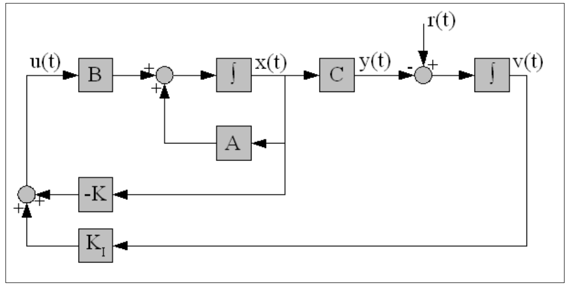
Tuning

- γ is the additional tuning parameter and influences the strength of the integral action. Don't push it too high (oscillations)!

Procedure

The procedure is similar to the standard LQR, but applied on an augmented state space system. The augmented matrices are denoted as $\tilde{A}, \tilde{B}, \tilde{\bar{C}}, \tilde{Q}, \tilde{K}, \tilde{\Phi}$.

Solution



Augmented system:

$$\tilde{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\tilde{\bar{C}} = \begin{bmatrix} \bar{C} & 0 \\ 0 & \gamma \end{bmatrix}$$

$$\tilde{Q} = \tilde{\bar{C}}^T \cdot \tilde{\bar{C}}$$

The solution is then found by solving the Ricatti equations on the augmented system:

$$\tilde{\Phi} \cdot \tilde{B} \cdot R^{-1} \cdot \tilde{B}^T \cdot \tilde{\Phi} - \tilde{\Phi} \cdot \tilde{A} - \tilde{A}^T \cdot \tilde{\Phi} - \tilde{Q} = 0$$

$$\tilde{K} = R^{-1} \cdot \tilde{B}^T \cdot \tilde{\Phi}$$

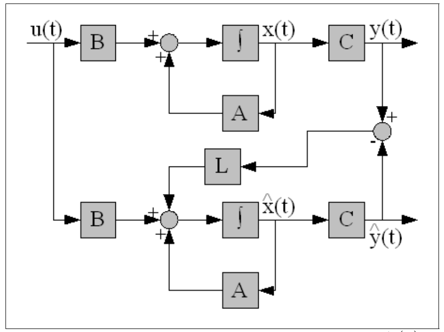
And then decomposing the resulting $\tilde{K}$ as follows: $[K \quad -K_I] = \tilde{K}$

4 Observer(LQG)

Aim

The aim of the observer is to estimate the state variables of the system if these are not directly available.

Task



Find the gain matrix L such that the estimated state $\hat{x}(t)$ is as close as possible to the real state $x(t)$. μ can be used as tuning parameter and has not really got any physical interpretation.

Solution

- Solve the dual Ricatti equation for the positiv-definit matrix Ψ :
  $\frac{1}{\mu} \Psi \cdot C^T \cdot C \cdot \Psi - \Psi \cdot A^T - A \cdot \Psi - B \cdot B^T = 0$
- Compute K:

$$L=(\frac{1}{\mu}\cdot C\cdot\Psi)^T$$

Tuning

- Smaller choices of μ generally lead to faster observers
- Often a good approach is to choose the observer poles($A-LC$) 3 times faster than the system's($A-BK$).
- If the observer is chosen too fast it can become instable to modeling errors or noise.

Conditions

- (A,B) must be controllable
- (A,C) must be observable.

Error Dynamics

This is the dynamic of the error between $\hat{x}(t)$ and $x(t)$:

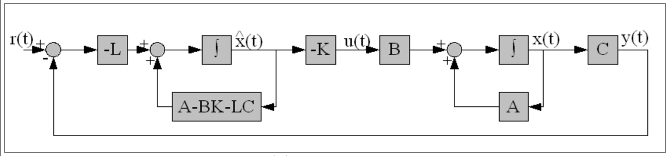
$$e(t)=x(t)-\hat{x}(t)$$
$$\dot{e}(t)=(A-LC)e(t)$$

The system matrix $A-LC$ has always negativ eigenvalues and therefore the error $e(t)$ converges to 0.

5 Output Feedback(OF), LTR

5.1 Simple Output Feedback

Trough combining an observer with a state feedback controller and reordering the blocks we obtain the following output feedback controller:



Note: the reference signal $r(t)$ was included in this diagram. In general the resulting control system will have some important steady state error (LQR is designed to bring the state to zero). To a good reference tracking behavior go to chapter 6.

Open-Loop

Open-loop state space matrices:

$$A_{LTR,OL}=\begin{bmatrix}A & -BK\\0 & A-BK-LC\end{bmatrix}\qquad B_{LTR,OL}=\begin{bmatrix}0\\-L\end{bmatrix}$$
$$C_{LTR,OL}=\begin{bmatrix}C & 0\end{bmatrix}$$

Loop gain:

$$L_{LTR}=C\cdot(sI-A)^{-1}\cdot B\cdot K\cdot(sI-(A-BK-LC))^{-1}\cdot L$$

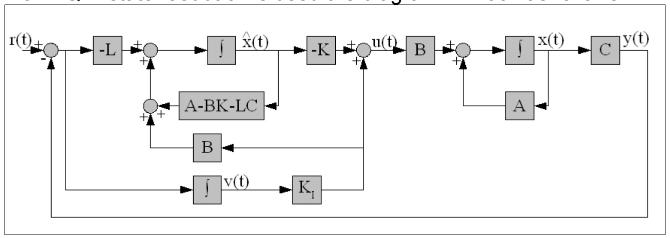
Closed-Loop

Closed-loop state space matrices:

$$A_{LTR,CL}=\begin{bmatrix}A & -BK\\LC & A-BK-LC\end{bmatrix}\qquad B_{LTR,CL}=\begin{bmatrix}0\\-L\end{bmatrix}$$
$$C_{LTR,CL}=\begin{bmatrix}C & 0\end{bmatrix}$$

5.2 Output feedback with Integral action

If an LQRI state feedback is used the diagram will look as follows:



This system already possesses good reference tracking qualities, so feedforward action like in chapter 6 is not absolutely necessary

Open-Loop

Open-loop state space matrices:

$$A_{LTR,OL}=\begin{bmatrix}A & -BK & BK_I\\0 & A-BK-LC & BK_I\\0 & 0 & 0\end{bmatrix}\qquad B_{LTR,OL}=\begin{bmatrix}0\\-L\\I\end{bmatrix}$$
$$C_{LTR,OL}=\begin{bmatrix}C & 0 & 0\end{bmatrix}$$

Closed-Loop

Closed-loop state space matrices:

$$A_{LTR,CL}=\begin{bmatrix}A & -BK & BK_I\\LC & A-BK-LC & BK_I\\C & 0 & 0\end{bmatrix}\qquad B_{LTR,CL}=\begin{bmatrix}0\\-L\\I\end{bmatrix}$$
$$C_{LTR,CL}=\begin{bmatrix}C & 0 & 0\end{bmatrix}$$

5.3 Loop Transfer Recovery(LTR)

Using an observer in state of the real state $x(t)$ strongly reduces the robustness of the controller. By iterating on the tuning parameters, part of the robustness can be reobtained.
LTR is called the procedure to design iteratively the LQR and LQG. The standard procedure looks as follows:

- Design a state feedback controller for your plant. Adapt the parameters Q and R until your specifications are met.
- Design an observer. Iterate on the parameter γ, until your specification are recovered to your satisfaction. In general it is advisable to begin with a relatively large value and decrease until your open-loop and closed-loop behaviors are satisfying.

6 Feedforward Action, Reference Tracking

Reference Tracking

The LQR/LQG approach leads in general to systems with very nice stabilizing qualities. However it does not guarantee for any good reference tracking.
There are two solutions if reference tracking problems are occuring:

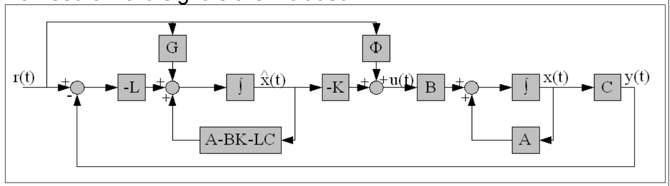
- use an LQRI state feedback controller (chapter 3)
- introduce some feedforward action to improve the reference tracking

Error dynamic

The feedforward action can also be used to make the error dynamik between $x(t)$ and $\hat{x}(t)$ independent of changes in the reference value $r(t)$.

Standard Feedforward Action

To Feedforward signals are included:



To obtain good reference tracking and error independence, the matrix $G\in\mathbb{R}^{n\times m}$ and $\Phi\in\mathbb{R}^{m\times m}$ can be computed as follows:

$$\Phi=-[C^*\cdot A^*\cdot B^*]^{-1}$$
$$G=L+B\Phi$$

with:

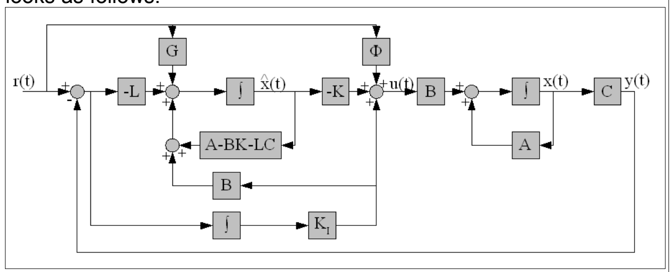
$$A^*=\begin{bmatrix}A & -BK\\LC & A-BK-LC\end{bmatrix}\qquad B^*=\begin{bmatrix}B\\B\end{bmatrix}$$
$$C^*=\begin{bmatrix}C & 0\end{bmatrix}$$

The resulting closed-loop behavior has the same matrices:

$$A_{CL}=A^*,\quad B_{CL}=B^*,\quad C_{CL}=C^*$$

Feedforward with integral action

In the case that integral action was included (LQRI), the diagram looks as follows:



Here the computation of both matrices simplifies to:

$$\Phi=I$$
$$G=L+B$$

And the resulting closed-loop state space representation is:

$$A_{CL}=\begin{bmatrix}A & -BK & BK_I\\LC & A-BK-LC & BK_I\\C & 0 & 0\end{bmatrix}\qquad B_{CL}=\begin{bmatrix}B\\B\\I\end{bmatrix}$$
$$C_{CL}=\begin{bmatrix}C & 0 & 0\end{bmatrix}$$

Note: This feedforward action can only be applied if the number of inputs is equal to the number of outputs!