LET US REMEMBER:

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad (1)$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

But does it make sense to talk about the sum of infinitely many terms?

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21, . . . and, after the nth term, we get $n(n + 1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots+\frac{1}{2^n}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots=1$$

We use a similar idea to determine whether or not a general series (1) has a sum. We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Convergence of a series

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges to the sum** s, and we write

$$\sum_{n=1}^{\infty} a_n = s,$$

if $\lim_{n \to \infty} s_n = s$, where s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{j=1}^{n} a_j.$$

THEOREM **The alternating series test**

Suppose $\{a_n\}$ is a sequence whose terms satisfy, for some positive integer N,

(i) $a_n a_{n+1} < 0$ for $n \geq N$,

(ii) $|a_{n+1}| \leq |a_n|$ for $n \geq N$, and

(iii) $\lim_{n\to\infty} a_n = 0$,

that is, the terms are ultimately alternating in sign and decreasing in size, and the sequence has limit zero. Then the series $\sum_{n=1}^{\infty} a_n$ converges.

THEOREM **Error estimate for alternating series**

If the sequence $\{a_n\}$ satisfies the conditions of the alternating series test so that the series $\sum_{n=1}^{\infty} a_n$ converges to the sum s, then the error in the approximation $s \approx s_n$ (where $n \geq N$) has the same sign as the first omitted term $a_{n+1} = s_{n+1} - s_n$, and it size is no greater than the size of that term:

$$|s - s_n| \leq |s_{n+1} - s_n| = |a_{n+1}|.$$

How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+2^n}$ are needed to compute the sum of the series with error less than 0.001?

Solution This series satisfies the hypotheses for Theorem . If we use the partial sum of the first n terms of the series to approximate the sum of the series, the error will satisfy

$$|\text{error}| \leq |\text{first omitted term}| = \frac{1}{1+2^{n+1}}.$$

This error is less than 0.001 if $1 + 2^{n+1} > 1{,}000$. Since $2^{10} = 1{,}024$, $n + 1 = 10$ will do; we need 9 terms of the series to compute the sum to within 0.001 of its actual value.

The following example shows how the error bound associated with the alternating series test can also be used for such approximations. When the terms a_n of a series (i) alternate in sign, (ii) decrease steadily in size, and (iii) approach zero as $n \to \infty$, then the error involved in using a partial sum of the series as an approximation to the sum of the series has the same sign as, and is no greater in absolute value than, the first omitted term.

Find $\cos 43°$ with error less than $1/10{,}000$.

Solution We give two alternative solutions:

METHOD I. We can use the Maclaurin series for cosine:

$$\cos 43° = \cos\frac{43\pi}{180} = 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^4 - \cdots.$$

Now $43\pi/180 \approx 0.75049\cdots < 1$, so the series above must satisfy the conditions (i)–(iii) mentioned above. If we truncate the series after the nth term

$$(-1)^{n-1}\frac{1}{(2n-2)!}\left(\frac{43\pi}{180}\right)^{2n-2},$$

then the error E will be bounded by the size of the first omitted term:

$$|E| \le \frac{1}{(2n)!}\left(\frac{43\pi}{180}\right)^{2n} < \frac{1}{(2n)!},$$

The error will not exceed $1/10{,}000$ if $(2n)! > 10{,}000$, so $n = 4$ will do $(8! = 40{,}320)$.

$$\cos 43^\circ \approx 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^{2} + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^{4} - \frac{1}{6!}\left(\frac{43\pi}{180}\right)^{6} \approx 0.731\,35\cdots$$

METHOD II. Since 43° is close to $45° = \pi/4$ rad, we can do a bit better by using the Taylor series about $\pi/4$ instead of the Maclaurin series:

$$\begin{aligned}\cos 43° &= \cos\left(\frac{\pi}{4} - \frac{\pi}{90}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{90} + \sin\frac{\pi}{4}\sin\frac{\pi}{90}\\ &= \frac{1}{\sqrt{2}}\left[\left(1 - \frac{1}{2!}\left(\frac{\pi}{90}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{90}\right)^4 - \cdots\right)\right.\\ &\qquad \left. + \left(\frac{\pi}{90} - \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 + \cdots\right)\right].\end{aligned}$$

Since

$$\frac{1}{4!}\left(\frac{\pi}{90}\right)^4 < \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 < \frac{1}{20{,}000},$$

we need only the first two terms of the first series and the first term of the second series:

$$\cos 43° \approx \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{90} - \frac{1}{2}\left(\frac{\pi}{90}\right)^2\right) \approx 0.731\,358\cdots.$$

(In fact, $\cos 43° = 0.731\,353\,7\cdots$.)

Find the Maclaurin series for

$$E(x) = \int_0^x e^{-t^2}\,dt,$$

and use it to evaluate $E(1)$ correct to 3 decimal places.

Solution The Maclaurin series for $E(x)$ is given by

$$\begin{aligned} E(x) &= \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots\right) dt \\ &= \left(t - \frac{t^3}{3} + \frac{t^5}{5 \times 2!} - \frac{t^7}{7 \times 3!} + \frac{t^9}{9 \times 4!} - \cdots\right)\Bigg|_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{5 \times 2!} - \frac{x^7}{7 \times 3!} + \frac{x^9}{9 \times 4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}, \end{aligned}$$

and is valid for all x because the series for e^{-t^2} is valid for all t. Therefore,

$$E(1) = 1 - \frac{1}{3} + \frac{1}{5 \times 2!} - \frac{1}{7 \times 3!} + \cdots$$

$$\approx 1 - \frac{1}{3} + \frac{1}{5 \times 2!} - \frac{1}{7 \times 3!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)(n-1)!}.$$

We stopped with the nth term. Again, the alternating series test assures us that the error in this approximation does not exceed the first omitted term, so it will be less than 0.0005, provided $(2n+1)n! > 2{,}000$. Since $13 \times 6! = 9{,}360$, $n = 6$ will do. Thus,

$$E(1) \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1{,}320} \approx 0.747,$$

rounded to 3 decimal places.

Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

The Fundamental Theorem of Calculus gives

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \cdots$$

$$\approx 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} \approx 0.7475$$

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$$

Estimate $\int_0^1 \sin x^2\,dx$ with an error of less than 0.001.

Solution

$$\int_0^1 \sin x^2\,dx = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} - \cdots.$$

The series alternates, and we find by experiment that

$$\frac{1}{11\cdot 5!} \approx 0.00076$$

is the first term to be numerically less than 0.001. The sum of the preceding two terms gives

$$\int_0^1 \sin x^2\,dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

With two more terms we could estimate

$$\int_0^1 \sin x^2\,dx \approx 0.310268$$

with an error of less than 10^{-6}. With only one term beyond that we have

$$\int_0^1 \sin x^2\,dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.310268303,$$

with an error of about 1.08×10^{-9}.

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

We represent $\ln x$ as a Taylor series in powers of $x - 1$.

$$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \cdots,$$

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \left(1 - \frac{1}{2}(x - 1) + \cdots\right) = 1.$$

Evaluate

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}.$$

Solution The Taylor series for $\sin x$ and $\tan x$, to terms in x^5, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Hence,

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

and

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$= -\frac{1}{2}.$$

Find the maclaurin series of the following function

$f(x) = \sqrt{1 + \sin x}$.

$\sin x = 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}$

$1 = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}$

$+$

$1 + \sin x = (\sin \frac{x}{2} + \cos \frac{x}{2})^2$

$\sqrt{1+\sin x} = \sin \frac{x}{2} + \cos \frac{x}{2}$

$\sin \frac{x}{2} = \frac{x}{2} - \frac{x^3}{2^3 . 3!} + \frac{x^5}{2^5 . 5!} - \cdots$

$\cos \frac{x}{2} = 1 - \frac{x^2}{2^2 . 2!} + \frac{x^4}{2^4 . 4!} - \cdots$

$+$

$f(x) = 1 + \frac{x}{2} - \frac{x^2}{2^2 . 2!} - \frac{x^3}{2^3 . 3!} + \frac{x^4}{2^4 . 4!} + \frac{x^5}{2^5 . 5!} - \cdots$

It is convergent for all x .