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Research Article

Global Bifurcation of Fourth-Order Nonlinear Eigenvalue Problems' Solution

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In this paper, we study the global bifurcation of infinity of a class of nonlinear eigenvalue problems for fourth-order ordinary differential equations with nondifferentiable nonlinearity. We prove the existence of two families of unbounded continuance of solutions bifurcating at infinity and corresponding to the usual nodal properties near bifurcation intervals.

1. Introduction

We consider the following nonlinear eigenvalue problem:

$$\ell y \equiv (p(x)y'')'' - (q(x)y')' = \lambda r(x)y + h\left(x, y, y', y'', y''', \lambda\right), \quad x \in (0,1), \tag{1}$$

$$\begin{aligned} &y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0,\\ &y(0)\cos\beta + Ty(0)\sin\beta = 0,\\ &y'(1)\cos\gamma + (py'')(1)\sin\gamma = 0,\\ &y(1)\cos\delta - Ty(1)\sin\delta = 0, \end{aligned} \tag{2}$$

where $\lambda \in \mathbb{R}$ is a real parameter, $Ty \equiv (py'')' - qy'$, p is a positive twice continuously differentiable on $[0,1]$, q is a positive continuously differentiable function on $[0,1]$, r is a nonnegative continuous function on $[0,1]$ such that $r(x) \not\equiv 0$ on any subinterval of $[0,1]$, α, β, γ, and δ are any real numbers between 0 and $\pi/2$. However, $\alpha = \gamma = 0$, $\beta = \delta = (\pi/2)$ cannot be the case; likewise, α, β, γ, and δ variables cannot take the value $\pi/2$ values at the same time. The function h is represented as $h = af + g$, where $f, g \in C([0,1] \times \mathbb{R}^5)$ and satisfy the following conditions: there exists $\underline{f_0}$ and $\overline{f_0}$ such that

$$\begin{aligned} \underline{f_0} &= \liminf_{|u|\longrightarrow 0} \frac{f(x,u,s,\lambda)}{u},\\ \overline{f_0} &= \limsup_{|u|\longrightarrow 0} \frac{f(x,u,s,\lambda)}{u}, \end{aligned} \tag{3}$$

uniformly in $x \in [0,1]$, $s \in \mathbb{R}$, $|s| \le m_0$, $\lambda \in \mathbb{R}$, where m_0 is some small positive constant;

$$\begin{aligned} g(x,u,s,v,w,\lambda) &= o(|u| + |s| + |v| + |w|)\\ &\text{as } |u| + |s| + |v| + |w| \longrightarrow 0, \end{aligned} \tag{4}$$

uniformly in $(x,\lambda) \in [0,1] \times \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$.

Problems (1)-(2) occur in the study of various processes in mechanics and physics (see [1] and [2]). Particularly, it arises when solving an important problem of aeroelasticity, the loss of stability of a flexible elongated plate under the action of a supersonic gas flow, compressed or stretched by external stresses (see [1] and [3]).

Problems (1)-(2) were considered in [4] (see also [5]) in the case when $r(x)$ is strictly positive on $[0,1]$, $h = f + g$, and f satisfies the condition

$$\left|\frac{f(x,u,s,v,w,\lambda)}{u}\right| \le M, \quad x \in [0,1],\ (u,s,v,w) \in \mathbb{R}^4, u \ne 0,$$
$$|u| + |s| + |v| + |w| \le m_0, \quad \lambda \in \mathbb{R}, \tag{5}$$

where M is a positive constant. In [4], a global bifurcation result for problems (1)-(2) under these conditions was obtained. The purpose of this paper is to seek an answer to the question that "what happens if the function $r(x)$ is not strictly positive in the range $[0,1]$?"

Several nonlinear eigenvalue problems for the Sturm–Liouville equation are considered in the literatures [6–10] and the bibliography therein. In these papers, the global bifurcation results were obtained, namely, the unbounded continuance of solutions bifurcating from intervals of the line of trivial solutions and possessing the usual nodal properties was established.

The structure of this paper is as follows. In Section 2, we study the structure of root subspaces and the oscillatory properties of the eigenfunctions of the linear problem obtained from (1)-(2) by setting $h \equiv 0$. Moreover, we establish a global bifurcation result for problems (1)-(2) for $f \equiv 0$. In Section 3, using an approximate problem, we prove the existence of solutions of problems (1)-(2) with small norms and usual nodal properties. Next, we find the bifurcation intervals of the line of trivial solutions concerning the sets with fixed oscillation count. Finally, we establish the existence of global continue of solutions bifurcating from these intervals.

2. Preliminary

By B.C., denote the set of functions that satisfy the boundary conditions (2) and consider the following linear eigenvalue problem:

$$\ell y = \lambda r(x) y, \quad x \in (0,1),\ y \in \text{B.C.}, \tag{6}$$

obtained from (1)-(2) by setting $h \equiv 0$. Let (λ, y) be an eigenpair of problem (6). By multiplying both sides of the equation in (1) by $\overline{y}$, integrating the resulting equality from 0 to 1, we get

$$\int_0^1 \left\{p(x)\left|y''(x)\right|^2 + q(x)\left|y'(x)\right|^2\right\}dx + N[|y|] = \lambda \int_0^1 r(x)|y(x)|^2 dx, \tag{7}$$

where

$$N[y] = \left[y'(0)\right]^2 \cot\alpha + \left[y(0)\right]^2 \cot\beta + \left[y'(1)\right]^2 \cot\gamma + \left[y(1)\right]^2 \cot\delta. \tag{8}$$

Since $r \in C([0,1])$, $r(x) \ge 0$ for $x \in [0,1]$ and $r(x) \not\equiv 0$ on any subinterval of $[0,1]$, and $\alpha, \beta, \gamma, \delta \in [0, (\pi/2)]$, except for the cases $\alpha = \gamma = 0$, $\beta = \delta = (\pi/2)$, and $\alpha = \beta = \gamma = \delta = (\pi/2)$ by (8). it follows from (7) that $\lambda > 0$ and all eigenvalues of the problem (6) are positive. Hence, by following the arguments of Banks and Kurowski [11], we can justify the following result.

Theorem 1. *The eigenvalues of the linear spectral problem (6) are positive and simple and form an unboundedly increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. The eigenfunction $y_k(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue λ_k, has exactly $k-1$ simple zeros in $(0,1)$. Moreover, if (i) $y_k(0)y_k'(0) > 0$, (ii) $y_k(0) = 0$, or (iii) $y_k'(0) = 0$ and $y_k(0)y_k''(0) > 0$, then $y_k'(x)$, $k \in \mathbb{N}$, has either $k-1$ or k simple zeros in the interval $(0,1)$, if (iv) $y_k(0)y_k'(0) < 0$, (v) $y_k'(0) = 0$, and $y_k(0)y_k''(0) < 0$, or (vi) $y_k'(0) = y_k''(0) = 0$, then $y_1'(x)$ has no zeros and $y_k'(x)$ for $k \ge 2$ has either $k-1$ or $k-2$ simple zeros in $(0,1)$.*

Let E be a Banach space, $C^3[0,1] \cup \text{B.C.}$ with the usual norm $\|u\|_3 = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|u'''\|_\infty$, where $\|u\|_\infty = \max_{x\in[0,1]}|u(x)|$.

To preserve the nodal properties along the global continuance of solutions to problems (1)-(2), Aliyev [4] constructed the sets S_k^ν, $k \in \mathbb{N}$, $\nu \in \{+,-\}$, of functions of E which possess the nodal properties of eigenfunctions of the spectral problem (6) and their derivatives by using the Prüfer type transformation. It is obvious that for each $k \in \mathbb{N}$ and each $\nu \in \{+,-\}$, the sets S_k^+ and S_k^- are disjoint and open subsets of E.

Consider the following eigenvalue problem:

$$(\ell y)(x) + r(x)\psi(x)y(x) = \lambda r(x)y(x), \quad x \in (0,1), y \in \text{B.C.}, \tag{9}$$

where $\psi \in L_1[0,1]$.

Remark 1. By Theorem 1, it follows from Theorem 1.2 and Remark 4.1 in [4] that the eigenvalues of problem (9) are real and simple and form an unboundedly increasing sequence $\{\lambda_{k,\psi}\}_{k=1}^{\infty}$. Moreover, the eigenfunction $y_{k,\psi}(x)$ corresponding to the eigenvalue $\lambda_{k,\psi}$ lies in S_k.

Lemma 1 (see [4], Lemma 1.1). *If $(\lambda, y) \in \mathbb{R} \times E$ is a solution of (1)-(2) such that $(\lambda, y) \in \mathbb{R} \times \partial S_k^{\nu}$, then $y \equiv 0$.*

Let C denote the closure of the set of nontrivial solutions of the nonlinear problems (1)-(2) in $\mathbb{R} \times E$.

If $f \equiv 0$, then the following global bifurcation results for (1)-(2) are obtained.

Theorem 2. *For each $k \in \mathbb{N}$, there exist continua C_k^+ and C_k^- of C containing $(\lambda_k, 0)$ that are unbounded in $\mathbb{R} \times E$ and contained in $(\mathbb{R} \times S_k^+) \cap \{(\lambda_k, 0)\}$ and $(\mathbb{R} \times S_k^-) \cup \{(\lambda_k, 0)\}$, respectively.*

The proof can be solved similarly to Theorem 1.1 in [4] by using Theorem 1 and Lemma 1.

3. Global Bifurcation of Solutions of Problems (1)-(2)

Consider the following approximate problem:

$$\ell y = \lambda r(x)y + r(x)f\left(x, |y|_3^{\varepsilon}y, y', y'', y''', \lambda\right) + g\left(x, y, y', y'', y''', \lambda\right), \quad x \in (0,1), y \in \text{B.C.}, \tag{10}$$

where $\varepsilon > 0$.

If $|u| + |s|$ is sufficiently small, then by virtue of condition (3), we have

$$\frac{|f(x, |u|^{\varepsilon}u, s, v, w, \lambda)|}{|u| + |s| + |v| + |w|} = \frac{|f(x, |u|^{\varepsilon}u, s, v, w, \lambda)|}{|u|^{\varepsilon}|u|} \frac{|u|^{\varepsilon}|u|}{|u| + |s| + |v| + |w|} \leq \max\left\{\left|\underline{f_0} - 1\right|, \left|\overline{f_0} + 1\right|\right\} (|u| + |s| + |v| + |w|)^{\varepsilon}, \tag{11}$$

which is uniformly in $x \in [0,1]$ and $\lambda \in \mathbb{R}$. Hence,

$$r(x)f(x, |u|^{\varepsilon}u, s, v, w, \lambda) = o(|u| + |s| + |v| + |w|) \quad \text{as } |u| + |s| + |v| + |w| \longrightarrow 0, \tag{12}$$

is also uniformly in $(x, \lambda) \in [0,1] \times \Lambda$ for any bounded interval $\Lambda \subset \mathbb{R}$. Then, it follows from Theorem 2 that for each $k \in \mathbb{N}$, there exists unbounded continuances $C_{k,\varepsilon}^+$ and $C_{k,\varepsilon}^-$ of solutions of problem (10) such that

$$\begin{aligned} (\lambda_k, 0) &\in C_{k,\varepsilon}^+ \subset (\mathbb{R} \times S_k^+) \cup \{(\lambda_k, 0)\} \\ (\lambda_k, 0) &\in C_{k,\varepsilon}^- \subset (\mathbb{R} \times S_k^-) \cup \{(\lambda_k, 0)\}. \end{aligned} \tag{13}$$

Let

$$\begin{aligned} M &= \max\left\{\left|\underline{f_0}\right|, \left|\overline{f_0}\right|\right\}, \\ J_k &= [\lambda_k - M, \lambda_k + M], \\ J_k(\tau) &= [\lambda_k - M - \tau, \lambda_k + M + \tau], \\ I_k &= \left[\lambda_k - \overline{f_0}, \lambda_k - \underline{f_0}\right], \\ I_k(\tau) &= \left[\lambda_k - \overline{f_0} - \tau, \lambda_k - \underline{f_0} + \tau\right], \end{aligned} \tag{14}$$

and $\tau_0 > 0$ be some fixed sufficiently small number.

Lemma 2. *For every $k \in \mathbb{N}$, each $\nu \in \{+,-\}$, and for each sufficiently small $\kappa > 0$, there exists a solution $(\widetilde{\lambda}, \widetilde{y})$ of problems (1)-(2) such that*

$$\begin{aligned} &\widetilde{\lambda} \in J_k(\tau_0), \\ &\widetilde{y} \in S_k^{\nu}, \\ &\|\widetilde{y}\|_3 = \kappa. \end{aligned} \tag{15}$$

Proof. By the above arguments, for any $\varepsilon > 0$, problem (10) has a solution $(\lambda_\varepsilon, y_\varepsilon)$ such that

$$\begin{aligned} &y_\varepsilon \in S_k^{\nu}, \\ &\|y_\varepsilon\|_3 = \kappa. \end{aligned} \tag{16}$$

Let

$$\psi_\varepsilon(x) = \begin{cases} -\dfrac{f\left(x, |y_\varepsilon(x)|^{\varepsilon} y_\varepsilon(x), y_\varepsilon'(x), y_\varepsilon''(x), y_\varepsilon'''(x), \lambda_{\kappa,\varepsilon}^{+}\right)}{y_\varepsilon(x)}, & \text{if } y_\varepsilon(x) \neq 0, \\ 0, & \text{if } y_\varepsilon(x) \neq 0. \end{cases} \tag{17}$$

Then, $(\lambda_\varepsilon, y_\varepsilon)$ is a solution of the following nonlinear problem:

$$(\ell y)(x) + r(x)\psi_\varepsilon(x)y(x) = \lambda r(x)y(x) + g\left(x, y(x), y'(x), y''(x), y'''(x), \lambda\right), \tag{18}$$

where $x \in (0,1)$, $y \in$ B.C.. By condition (4), the nonlinear problem (18) is linearizable, and the linearization of this problem for $y = 0$ is given by

$$(\ell y)(x) + r(x)\psi_\varepsilon(x)y(x) = \lambda r(x)y(x), \quad x \in (0,1), y \in \text{B.C.}. \tag{19}$$

In (3), there exists $\kappa_0 \in (0, m_0)$ such that for any $(y,s) \in \mathbb{R}^2$, $|y| + |s| < \kappa_0$ and $\lambda \in \mathbb{R}$, the following inequality holds:

$$\underline{f_0} - \frac{\tau_0}{2} < \frac{f(x,y,s,\lambda)}{y} < \overline{f_0} + \frac{\tau_0}{2}. \tag{20}$$

Let $\kappa < \kappa_0$, then

$$\left|\psi_\varepsilon(x)\right| \leq M_0 |y(x)|^{\varepsilon} \leq M_0, \quad x \in [0,1], \tag{21}$$

where $M_0 = M + (\tau_0/2)$, follows from (18).

By virtue of the maximum and minimum property of eigenvalues (see [2], Ch. 6), the k-th eigenvalue $\lambda_{k,\psi_\varepsilon}$ of (19) is determined from the relation

$$\lambda_{k,\psi_\varepsilon} = \max_{y \in V^{(k-1)}} \min_{y \in \text{B.C.}} \left\{ \frac{\int_0^1 \left\{ p(x) y''^2(x) + q(x) y'^2(x) \right\} dx + N[y] + \int_0^1 r(x)\psi_\varepsilon(x) y^2(x) dx}{\int_0^1 r(x) y^2(x) dx} : \int_0^1 r(x)\varphi(x)y(x)dx = 0, \quad \varphi(x) \in V^{(k-1)} \right\}, \tag{22}$$

where $V^{(k-1)}$ is an arbitrary set of linearly independent functions $\varphi_i(x)$, $i = 1, 2, \ldots, k-1$. Hence,

$$\lambda_k = \max_{y \in V^{(k-1)}} \min_{y \in \text{B.C.}} \left\{ \frac{\int_0^1 \left\{ p(x) y''^2(x) + q(x) y'^2(x) \right\} dx + N[y]}{\int_0^1 r(x) y^2(x) dx} : \int_0^1 r(x)\varphi(x)y(x)dx = 0, \quad \varphi(x) \in V^{(k-1)} \right\}, \tag{23}$$

by (21) and (23), it follows from (22) that

$$\lambda_k - M_0 \leq \lambda_{k,\psi_\varepsilon} \leq \lambda_k + M_0, \tag{24}$$

i.e.,

$$\lambda_{k,\psi_\varepsilon} \in J_k\left(\frac{\tau_0}{2}\right). \tag{25}$$

Since $\|y_\varepsilon\|_3 = \kappa$ and $y_\varepsilon \in S_k^{\nu}$ by Theorem 2 (also from [12], Ch. 4), we can choose κ_0 as

$$\left|\lambda_\varepsilon - \lambda_{k,\psi_\varepsilon}\right| < \frac{\tau_0}{2} \quad \text{for } \kappa < \kappa_0. \tag{26}$$

From (25) and (26), we obtain

$$\lambda_\varepsilon \in J_k(\tau_0). \tag{27}$$

By conditions (3) and (4), it follows from (1) that $\{(\lambda_\varepsilon, y_\varepsilon):\ \varepsilon \in (0,1)\}$ is bounded in $\mathbb{R} \times C^4([0,1])$. Thus, we can find a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,1)$, $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$, such that $(\lambda_{\varepsilon_n}, y_{\varepsilon_n})$ converges to a solution $(\widetilde{\lambda}, \widetilde{y})$ of the nonlinear eigenvalue problems (1)-(2). Then, by using (16) and (27), we get $\widetilde{\lambda} \in J_k(\tau_0)$, $\|\widetilde{y}\|_3 = \kappa$, and $\widetilde{y} \in \overline{S_k^\nu} = S_k^\nu \cup \partial S_k^\nu$.

Consequently, it follows from Lemma 1 that $\widetilde{y} \in S_k^\nu$. The proof of this lemma is complete.

If there exists a sequence $\left\{(\lambda_{k,n}^\nu, y_{k,n}^\nu)\right\}_{n=1}^{\infty} \subset \mathbb{R} \times S_k^\nu$ of solutions to this problem which converges to $(\lambda, 0)$ in $\mathbb{R} \times E$ as $n \longrightarrow \infty$, then $(\lambda, 0) \in \mathbb{R} \times E$ is called a bifurcation point of problems (1)-(2) with respect to the set $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$. □

Corollary 1. *For every $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$, the set of bifurcation points of problems (1)-(2) with respect to the set $\mathbb{R} \times S_k^\nu$ is nonempty.*

Proof. Let $\{\kappa_n\}_{n=1}^{\infty} \subset (0, \kappa_0)$ be a sequence converging to zero as $n \longrightarrow \infty$, then by Lemma 2, for every $k \in \mathbb{N}$, each $\nu \in \{+, -\}$, and any $n \in \mathbb{N}$, there exists a solution $(\lambda_{k,n}^\nu, y_{k,n}^\nu)$ of (1)-(2) such that

$$\begin{aligned} \lambda_{k,n}^\nu &\in J_k(\tau_0), \\ y_{k,n}^\nu &\in S_k^\nu, \\ \left\|y_{k,n}^\nu\right\|_3 &= \kappa_n. \end{aligned} \tag{28}$$

Then, from the sequence $\left\{(\lambda_{k,n}^\nu, y_{k,n}^\nu)\right\}_{n=1}^{\infty}$, one can select a subsequence $\left\{(\lambda_{k,n_l}^\nu, y_{k,n_l}^\nu)\right\}_{l=1}^{\infty}$ which converges to $(\lambda, 0)$, where $\lambda \in J_k(\tau_0)$. This completes the proof. □

Lemma 3. *Let $(\lambda, 0)$ be a bifurcation point of (1)-(2) with respect to the set $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+, -\}$, then $\lambda \in I_k$.*

Proof. Suppose that $(\lambda, 0)$ is a bifurcation point of (1)-(2) with respect to the set $\mathbb{R} \times S_k^\nu$, then there exists a sequence $\left\{(\lambda_{k,n}^\nu, y_{k,n}^\nu)\right\}_{n=1}^{\infty}$ which converges to $(\lambda, 0)$ in $\mathbb{R} \times E$.

Let we assume $\lambda \notin I_k$. If we denote

$$\epsilon_0 = \frac{\operatorname{dist}\{\lambda, I_k\}}{2}, \tag{29}$$

then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the following inequality holds:

$$\left|\lambda_{k,n}^\nu - \lambda\right| < \epsilon_0. \tag{30}$$

Therefore,

$$\operatorname{dist}\left\{\lambda_{k,n}^\nu, I_k\right\} > \epsilon_0 \quad \text{for } n \geq n_0. \tag{31}$$

We leave $k \in \mathbb{N}$ and $\nu \in \{+, -\}$ arbitrary and fixed. It is evident that for each $n \in \mathbb{N}$, the pair $(\lambda_{k,n}^\nu, y_{k,n}^\nu)$ is a solution to the problem;

$$(\ell y)(x) + r(x)\psi_n(x)y(x) = \lambda r(x)y(x) + g\left(x, y(x), y'(x), y''(x), y'''(x), \lambda\right), \quad x \in (0,1), y \in \text{B.C.}, \tag{32}$$

where

$$\psi_n(x) = \begin{cases} -\dfrac{f\left(x, y_{k,n}^\nu(x), \left(y_{k,n}^\nu\right)'(x), \left(y_{k,n}^\nu\right)''(x), \left(y_{k,n}^\nu\right)'''(x), \lambda_{k,n}^\nu\right)}{y_{k,n}^\nu(x)}, & \text{if } y_{k,n}^\nu(x) \neq 0, \\ 0, & \text{if } y_{k,n}^\nu(x) \neq 0. \end{cases} \tag{33}$$

By condition (3), there exists $\widetilde{\kappa}_0 \in (0, m_0)$ such that for any $(y, s) \in \mathbb{R}^2$, $|y| + |s| < \widetilde{\kappa}_0$, and $\lambda \in \mathbb{R}$, the inequality

$$\underline{f_0} - \frac{\epsilon_0}{2} < \frac{f(x, y, s, \lambda)}{y} < \overline{f_0} + \frac{\epsilon_0}{2}, \tag{34}$$

holds. It is acceptable to take $n_1 \geq n_0$, thus

$$\left\|y_{k,n}^\nu\right\|_3 < \widetilde{\kappa}_0, \tag{35}$$

for all $n \geq n_1$. Then, from (34) and (33), we get

$$-\overline{f_0} - \frac{\epsilon_0}{2} < \psi_n(x) < -\underline{f_0} + \frac{\epsilon_0}{2}, \quad x \in [0,1], n \geq n_1. \tag{36}$$

By using (23) and replacing ψ_ε by ψ_n, we get

$$\lambda_k - \overline{f_0} - \frac{\epsilon_0}{2} < \lambda_{k,\psi_n} < \lambda_k - \underline{f_0} + \frac{\epsilon_0}{2} \quad \text{for } n \geq n_1, \tag{37}$$

$$\operatorname{dist}\left\{\lambda_{k,\psi_n}, I_k\right\} < \frac{\epsilon_0}{2} \quad \text{for } n \geq n_0, \tag{38}$$

where λ_{k,ψ_n} is the kth eigenvalue of the linear spectral problem

$$(\ell y)(x) + r(x)\psi_n(x)y(x) = \lambda r(x)y(x), \quad x \in (0,1), y \in \text{B.C.} \tag{39}$$

Note that Theorem 2 is applicable to the nonlinear problem (32) as a result of condition (4). Since $y_{k,n}^{\nu} \in S_k^{\nu}$ and $\|y_{k,n}^{\nu}\|_3 \longrightarrow 0$ as $n \longrightarrow \infty$, it follows from Theorem 2 that we can choose $n_2 \geq n_1$ so that

$$\left|\lambda_{k,n}^{\nu} - \lambda_{k,\psi_n}\right| < \frac{\epsilon}{2} \quad \text{for } n \geq n_2, \tag{40}$$

i.e.,

$$\operatorname{dist}\left\{\lambda_{k,n}^{\nu}, \lambda_{k,\psi_n}\right\} < \frac{\epsilon_0}{2} \quad \text{for } n \geq n_2, \tag{41}$$

Then, by (38) and (41), we get

$$\begin{aligned} &\operatorname{dist}\left\{\lambda_{k,n}, I_k\right\} \leq \operatorname{dist}\left\{\lambda_{k,n}, \lambda_{k,\psi_n}\right\} + \operatorname{dist}\left\{\lambda_{k,\psi_n}, I_k\right\} < \\ &\frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0 \quad \text{for } n \geq n_2, \end{aligned} \tag{42}$$

which contradicts relation (31). This completes the proof.

For each $k \in \mathbb{N}$ and each ν, we denote by $\widetilde{D}_k^{\nu}$ the union of all components $D_{k,\lambda}^{\nu}$ of C bifurcating from points $(\lambda, 0) \in I_k \times \{0\}$ with respect to the set $\mathbb{R} \times S_k^{\nu}$. Note that the set $\widetilde{D}_k^{\nu}$ is nonempty as a result of Corollary 1 and Lemma 3. It is obvious that the set $D_k^{\nu} = \widetilde{D}_k^{\nu} \cup (J_k \times \{0\})$ is connected in $\mathbb{R} \times E$. □

Theorem 3. *For each $k \in \mathbb{N}$ and each ν, the set D_k^{ν} is unbounded in $\mathbb{R} \times E$ and lies in $(\mathbb{R} \times S_k^{\nu}) \cup (I_k \times \{0\})$.*

The proof can be shown similar to Theorem 1.3 in [4] by using Lemma 2, Corollary 1, and Lemma 3.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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