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ORIGINAL PAPER

Period-3 Orbits of Sequential Dynamical Systems and Their Relationship to Error-Correcting Codes over Finite Fields

Tuğçe Ulutaş[1] · **Mehmet Emin Köroğlu**[2]

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Abstract
A sequential dynamical system consists of the following data; a finite graph Y with vertex set $v_1, \ldots, v_n$, a state set, local update functions, and an update ordering σ. We study period-3 orbits of SDSs on the complete graph K_n with identical local functions. We prove that the maximum number θ_{n+1} of 3-cycles in the phase space equals $A_3(n, 4)$, the largest size of a ternary code of length n with minimum Hamming distance at least 4. Our approach reduces the problem to a clique number computation in an explicit graph and yields a direct correspondence with optimal ternary $(n, 4)$-codes. We also give field-agnostic necessary conditions for prime period-p orbits and discuss extensions over $\mathbb{F}_p$.

Keywords Sequential dynamical systems · Periodic orbit · Complete graph · Codes over finite fields

Mathematics Subject Classification 05C90 · 11C08 · 11T06 · 11T71 · 94B05

1 Introduction

Sequential dynamical systems (SDSs) were introduced as a framework for formalizing certain types of large-scale computer simulations as compositions of local maps. An SDS is defined by a set of local variables $v_1, v_2, \ldots, v_n$, a collection of local update functions, and a specified update order. In the literature, the relationship between SDSs and codes over binary fields was first established by Defant in [1]. He showed that

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These authors contributed equally to this work.

✉ Mehmet Emin Köroğlu
mkoroglu@yildiz.edu.tr

Tuğçe Ulutaş
tugce.ulutas@std.yildiz.edu.tr

[1] Department of Mathematics, Istanbul Bilgi University, Istanbul 34060, Eyüpsultan, Türkiye

[2] Department of Mathematics, Yildiz Technical University, Istanbul 34220, Esenler, Türkiye

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the maximum number of 2-cycles is equal to the maximum number of codewords in a binary code of length $n-1$ with minimum Hamming distance at least 3, and he provided an interpretation of the code sequence beyond its original definition. For further and more detailed information on SDSs, the reader is referred to [2].

In [3], Barrett, Mortveit, and Reidys characterized invertible SDSs, fixed points, and analyzed how many distinct SDSs can be obtained by reordering a given multiset of update functions. In [4], the same authors provided a combinatorial upper bound for the number of non-equivalent SDSs for a given graph, and computed this bound explicitly for certain classes of graphs. Results concerning sequentially updated cellular automata over arbitrary graphs can be found in [5]. For further interesting results on SDSs, the reader is referred to [6], which contains generalizations of some known results for SDSs with permutation update schedules, as well as open problems in the area.

A graph colored in exactly k colors is called a k-colored graph. In [7], an expression is given for the number of labeled k-colored graphs on a given number of nodes. For information on periodic orbits and fixed points, see [8]. Since linear sequential dynamical systems on digraphs exhibit much richer properties than those on undirected graphs, significant results can be found in [9]. There is a connection between the clique number and acyclic orientations, and [10–14] present useful results on bijections between certain equivalence classes of permutations and acyclic orientations, as well as on counting the number of acyclic orientations and dependent edges.

Finding upper bounds for the chromatic number of graphs with bounded maximal clique size and degree is a classical problem. In [15], relationships between the clique number, chromatic number, and degree of a graph are established. Since the maximum number of codewords in a binary code of length n with minimum distance at least 4 equals the clique number of a certain graph, understanding maximal cliques in the graph of simplex codes [16] is valuable.

Sequential dynamical systems are also studied under the notion of cycle equivalence of a class of finite graph dynamical systems, where equivalence is related to their periodic orbits. Further results can be found in [17–20]. In addition, [21] analyzes the representation of graph states in the phase space, showing that these states can be identified with a class of non-singular curves. The work [22] presents a proof of the existence of connected, undirected graphs with prescribed orbit structure and provides an explicit construction procedure for such graphs.

Basic notations in combinatorics and algebraic coding theory can be found in [23] and [24]. Finding a bound for the maximum size $A_q(n, m)$ of a code of length n and minimum Hamming distance m over a finite field $\mathbb{F}_q$ is an important problem, and a special case is discussed in [25].

A sequential dynamical system is a special case of a discrete dynamical system, and discrete dynamical systems have numerous applications in Boolean networks [26, 27]. In [28], a lower bound on the covering radius was obtained, which is met with equality. For methods to find and count simple cycles of a given length in directed and undirected graphs, [29] is a useful reference. The work [30] addresses the NP-complete problem for some simple classes of sequential dynamical systems.

In this paper, we focus on cycles of length $m \geq 3$ and their maximum possible number. We show that this maximum equals the maximum number of codewords in a

ternary code of length n with minimum distance at least 4, namely $A_3(n, 4)$, and we identify $\theta_{n+1} = A_3(n, 4)$.

Now, we will recall some basics about undirected graphs and SDSs. For further and more detailed information, the reader may consult Refs. [2, 31, 32]. A simple undirected graph is an ordered pair $Y = (V, E)$ where $V \neq \varnothing$ is a finite set of vertices and $E \subseteq \big\{\{u, v\} : u, v \in V,\ u \neq v\big\}$. Thus, edges are unordered pairs of distinct vertices i.e., there is no orientation, and the graph has no loops and no multiple edges.

- A clique in Y is a subset $C \subseteq V$ such that every two distinct vertices of C are adjacent; equivalently, the subgraph induced by C is complete.
- The clique number of Y, denoted $\psi(Y)$, is the maximum cardinality of a clique in Y, i.e.,
$$\psi(Y) = \max\{\, |C| : C \subseteq V,\ C \text{ is a clique in } Y \,\}.$$

Sequential dynamical systems consist of the following parts.

- An undirected simple graph Y with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$.
- A set of states A. We will use the set of states $A = \mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}$.
- A collection of vertex functions $\{f_{v_i}\}_{i=1}^n$. Each vertex v_i of Y is assigned its own vertex function $f_{v_i} : A^{d(v_i)+1} \to A$, where $d(v_i)$ denotes the degree of v_i.
- A permutation $\sigma \in S_n$. The permutation σ is known as the update order.

Let $s(u)$ denote the state of a vertex u. Also, let

$$X(u_i) = (s(u_{j_1}), s(u_{j_2}), \ldots, s(u_{j_t}), s(u_i), s(u_{j_{t+1}}), \ldots, s(u_{j_{d(v_i)}})).$$

The vector $(s(u_1), s(u_2), \ldots, s(u_n))$ is the system state of the SDS. So, if Y is a complete graph then $X(u_i)$ is the system state of the SDS. For each vertex function f_{v_i}, define $L_{u_i} : A^n \to A^n$ by

$$L_{u_i}(v_1, v_2, ..., v_n) = (v_1, v_2, \ldots, v_{i-1}, f_{u_i}(X(u_i)), v_{i+1}, \ldots, v_n).$$

Then if we combine these L_{u_i} functions with σ, we obtain the SDS map $T : A^n \to A^n$ given by

$$T = L_{u_{\sigma(n)}} \circ L_{u_{\sigma(n-1)}} \circ \ldots \circ L_{u_{\sigma(1)}}.$$

To obtain the i-th system state, define the map $G_i : A^n \to A^n$ for each $i \in 1, 2, ..., n$ is defined as

$$G_i = L_{u_{\sigma(i)}} \circ L_{u_{\sigma(i-1)}} \circ \ldots \circ L_{u_{\sigma(1)}}.$$

Given any SDS on a graph Y with vertex functions $\{f_{v_i}\}_{i=1}^n$ and update order σ, we denote its SDS map T by the triple $[Y, \{f_{v_i}\}_{i=1}^n, \sigma]$.

Example 1 In this example we work over $\mathbb{F}_2$. Consider the graph Y with vertices v_1, v_2, v_3, and v_4 shown in Fig. 1. We define an SDS over Y with update order $\sigma = 3412$

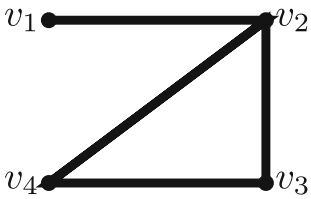


Fig. 1 The graph Y used in Example 1

and vertex functions f_{v_i}, $i = 1, 2, 3, 4$, given by

$$\begin{aligned} f_{v_1}(x_1, x_2) &= x_1 x_2, \\ f_{v_2}(x_1, x_2, x_3, x_4) &= x_1 x_4 + x_2 x_3, \\ f_{v_3}(x_1, x_2, x_3) &= x_1 + x_2 x_3 + 1, \\ f_{v_4}(x_1, x_2, x_3) &= x_1 + x_2 + x_3. \end{aligned}$$

Starting from the initial system state $(0, 1, 0, 1)$, we update the vertices v_1, v_2, v_3, and v_4 in the order specified by σ, using the adjacency of Y and the corresponding local update functions f_{v_1}, f_{v_2}, f_{v_3}, and f_{v_4}. Since $\sigma(1) = 3$, we first update vertex v_3 using f_{v_3} as

$$f_{v_3}\big(q(v_1), q(v_2), q(v_3)\big) = f_{v_3}(0, 1, 0) = 1.$$

Thus, v_3 changes to state 1. Since $\sigma(2) = 4$, we next update vertex v_4 using f_{v_4} as

$$f_{v_4}\big(q(v_1), q(v_2), q(v_3)\big) = f_{v_4}(0, 1, 1) = 0.$$

So, v_4 changes to state 0. Since $\sigma(3) = 1$, we then update vertex v_1 using f_{v_1} as

$$f_{v_1}\big(q(v_1), q(v_2)\big) = f_{v_1}(0, 1) = 0.$$

Hence, v_1 remains in state 0. Finally, since $\sigma(4) = 2$, we update vertex v_2 using f_{v_2} as

$$f_{v_2}\big(q(v_1), q(v_2), q(v_3), q(v_4)\big) = f_{v_2}(0, 1, 1, 0) = 1.$$

Thus, v_2 remains in state 1. Therefore, the resulting system state is

$$T(0, 1, 0, 1) = L_{u_{\sigma(4)}} \circ L_{u_{\sigma(3)}} \circ L_{u_{\sigma(2)}} \circ L_{u_{\sigma(1)}}(0, 1, 0, 1) = (0, 1, 1, 0).$$

Figure 2 illustrates how the vertices of the graph Y given in Fig. 1 are updated according to the update order $\sigma = 3412$ for the SDS described in Example 1, given the initial condition $(0, 1, 0, 1)$ and the specified local update functions f_{v_i}, $i = 1, 2, 3, 4$.

A useful tool for visualizing how T acts on the system's states is the phase space of the SDS. The phase space of an SDS is a directed graph and denoted$\Lambda(T)$. In Fig. 3 we give the phase space of the SDS in Example 1 showing all 16 states and their transitions with respect tothe update order $\sigma = 3412$. In this paper, we study the number of 3-cycles that can appear in the phase spaces of SDS defined over a complete graph K_n in which all vertexfunctions are the same.

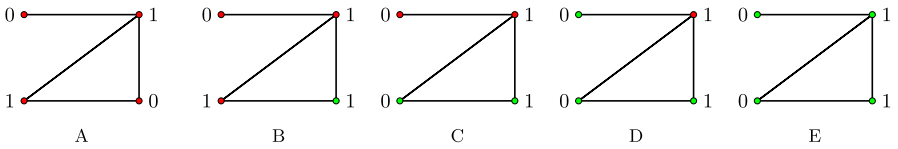


Fig. 2 The system update of the SDS given in Example 1 from the initial states $(v_1, v_2, v_3, v_4) = (0, 1, 0, 1)$. The first figure A displays the initial state of the SDS. The figures B-D illustrate intermediate steps in the system update. The final figure E shows the system state after the update is completed

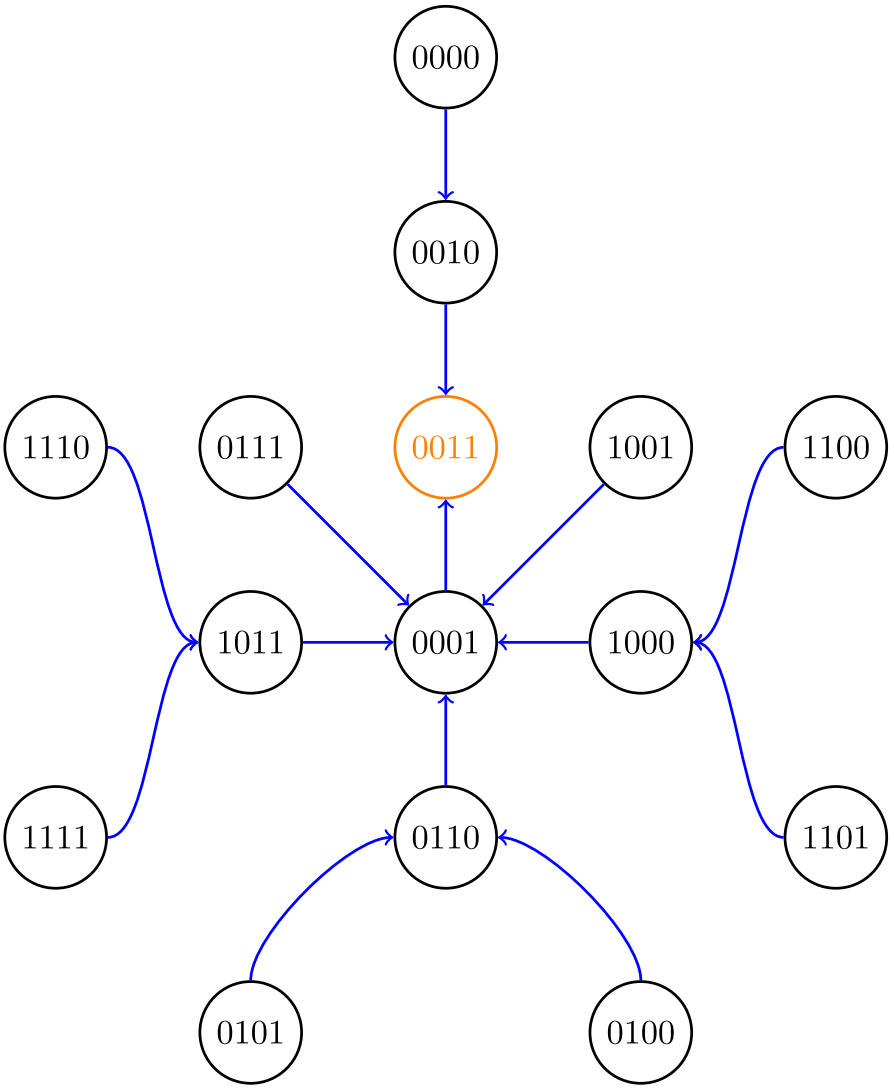


Fig. 3 (Color online) Phase space of the SDS in Example 1 showing all 16 states and their transitions with respect to the update order $\sigma = 3412$. To enhance the visual clarity of the image, we have omitted parentheses and commas from the vectors. For instance, the vector (0, 1, 0, 1) is represented as 0101

2 The Maximum Number of Periodic Orbits over Finite Fields

A phase space of an SDS can have many cycles of various lengths. In this section, we study the number of m-cycles where $m > 2$ that can appear in the phase spaces of SDS defined over a complete graph K_n in which all vertex functions are the same.

Definition 1 For a positive integer n, let

$$(\mathbb{F}_3)^{\mathbb{F}_3^n} := \{ h : \mathbb{F}_3^n \to \mathbb{F}_3 \}.$$

For $h \in (\mathbb{F}_3)^{\mathbb{F}_3^n}$ and $\sigma \in S_n$, let $\theta(h, \sigma)$ denote the number of 3-cycles in the phase space $\Lambda([K_n, h, \sigma])$. Define

$$\theta_n = \max_{h \in (\mathbb{F}_3)^{\mathbb{F}_3^n}} \theta(h, \mathrm{id}).$$

If W is a set and $T : W \to W$ is a function, then $\vec{\mathbf{w}} \in W$ is a periodic point of period 3 of T if $T^3(\vec{\mathbf{w}}) = \vec{\mathbf{w}}$ and $T(\vec{\mathbf{w}}) \neq \vec{\mathbf{w}}$, $T^2(\vec{\mathbf{w}}) \neq \vec{\mathbf{w}}$.

Lemma 1 *Let $n \geq 2$ be an integer. Let $f : A^n \to A$ be a function, and let $\sigma(1)\sigma(2)\ldots\sigma(n) \in S_n$ be a permutation. Suppose $\vec{\mathbf{u}} = (u_1, \ldots, u_n)$ is a periodic point of period 3 of the SDS map $T = [K_n, f, \sigma]$. Write $T(\vec{\mathbf{u}}) = \vec{\mathbf{w}} = (w_1, \ldots, w_n)$. For each $k \in 1, 2, \ldots, n$, we have $u_k \neq w_k$, and $T^2(\vec{\mathbf{u}}) = T(\vec{\mathbf{w}}) \neq \vec{\mathbf{u}}$. In particular, if $A = \mathbb{F}_3$, then T can be taken as $T(\vec{\mathbf{u}}) = \vec{\mathbf{u}} + \vec{2}$.*

Proof By the definition, $f(G_{i-1}(\vec{\mathbf{u}})) = w_{\sigma(i)}$. Also, $f(G_n(\vec{\mathbf{u}})) = f(T(\vec{\mathbf{u}})) = f(\vec{\mathbf{w}}) = f(G_0(\vec{\mathbf{w}})) = u_{\sigma(1)}$. $f(G_{j-1}(\vec{\mathbf{w}})) = u_{\sigma(j)}$ for any $j \in 1, 2, \ldots, n$. Then we can obtain that

$$f(G_n(\vec{\mathbf{w}})) = w_{\sigma(1)}.$$
$$f(G_n^2(\vec{\mathbf{u}})) = f(T^2(\vec{\mathbf{u}})) = f((T \circ T)(\vec{\mathbf{u}})) = f(T(\vec{\mathbf{w}})) = w_{\sigma(1)}.$$

$$f(G_n^2(\vec{\mathbf{w}})) = f(T^2(\vec{\mathbf{w}})) = f(T^2(T(\vec{\mathbf{u}}))) = f(T^3(\vec{\mathbf{u}})) = f(\vec{\mathbf{u}}) = f(G_0(\vec{\mathbf{u}})) = w_{\sigma(1)}.$$

$$\begin{array}{ll} f(G_{i-1}^2(\vec{\mathbf{u}})) = w_{\sigma(i+1)} & f(G_{i-1}^2(\vec{\mathbf{w}})) = u_{\sigma(i+1)} \\ f(G_i^2(\vec{\mathbf{u}})) = w_{\sigma(i+2)} & f(G_i^2(\vec{\mathbf{w}})) = u_{\sigma(i+2)} \\ f(G_{n-1}^2(\vec{\mathbf{u}})) = w_{\sigma(n+1)} & f(G_{n-1}^2(\vec{\mathbf{w}})) = u_{\sigma(n+1)} \end{array}$$

$$f(G_n^3(\vec{\mathbf{u}})) = f(T^3(\vec{\mathbf{u}})) = f(\vec{\mathbf{u}}) = f(G_0(\vec{\mathbf{u}})) = w_{\sigma(1)}.$$
$$f(G_n^3(\vec{\mathbf{w}})) = f(T^3(\vec{\mathbf{w}})) = f((T \circ T^3)(\vec{\mathbf{u}})) = f(T(\vec{\mathbf{u}})) = f(w).$$

$$\begin{array}{ll} f(G_{i-1}^3(\vec{\mathbf{u}})) = w_{\sigma(i+2)} & f(G_{i-1}^3(\vec{\mathbf{w}})) = u_{\sigma(i+2)} \\ f(G_i^3(\vec{\mathbf{u}})) = w_{\sigma(i+3)} & f(G_i^3(\vec{\mathbf{w}})) = u_{\sigma(i+3)} \\ f(G_{n-1}^3(\vec{\mathbf{u}})) = w_{\sigma(n+2)} & f(G_{n-1}^3(\vec{\mathbf{w}})) = u_{\sigma(n+2)} \end{array}$$

If $T(\vec{\mathbf{u}}) = \vec{\mathbf{u}}$, i.e., $u_{\sigma(i)} = w_{\sigma(i)}$. Then, we obtain $f(G_{i-1}(\vec{\mathbf{u}})) = f\,(G_{i-1}(\vec{\mathbf{u}}))$ and $f(G_{i-1}(\vec{\mathbf{w}})) = f\,(G_{i-1}(\vec{\mathbf{w}}))$. If we have this equality for some $i \in \{1, 2, \ldots, n\}$, then

$u_{\sigma(i+1) \pmod n} = w_{\sigma(i+1) \pmod n}$. This means that for some $i \in \{1, 2, \ldots, n\}$, $\vec{\mathbf{u}} = \vec{\mathbf{w}} = T(\vec{\mathbf{u}})$. But, by the assumption, we know that $\vec{\mathbf{u}}$ is a periodic point of T of period 3, so $T(\vec{\mathbf{u}}) \neq \vec{\mathbf{u}}$. So, $u_{\sigma(i)} \neq w_{\sigma(i)}$ for all $i \in \{1, 2, \ldots, n\}$. If $T^2(\vec{\mathbf{u}}) = T(T(\vec{\mathbf{u}})) = T(\vec{\mathbf{w}}) = T(w_1, \ldots, w_n) = (u_1, \ldots, u_n)$, then $G_n(w_1, \ldots, w_n) = (u_1, \ldots, u_n)$. This means that $G^2_{i-1}(u_1, \ldots, u_n) = G^2_i(u_1, \ldots, u_n)$ and $G^2_{i-1}(w_1, \ldots, w_n) = G^2_i(w_1, \ldots, w_n)$. By the previous equations, we can show that if $T(w_1, \ldots, w_n) = (u_1, \ldots, u_n)$ for some $i \in \{1, 2, \ldots, n\}$, then $(u_1, \ldots, u_n) = T^2(u_1, \ldots, u_n)$. But, by the assumption, we know that $(u_1, \ldots, u_n)$ is a periodic point of T of period 3 , so $T^2(u_1, \ldots, u_n) \neq (u_1, \ldots, u_n)$. Hence, $T(w_1, \ldots, w_n) \neq (u_1, \ldots, u_n)$ for all indices. □

The following corollary extends the statement of Lemma 1 to the case of an arbitrary finite field $\mathbb{F}_p$. The proof can be established in a similar manner.

Corollary 1 *Let $n \geq 2$ be an integer. Let $f : A^n \to A$ be a function, and let $\sigma(1)\sigma(2)\ldots\sigma(n) \in S_n$ be a permutation. Suppose $\vec{\mathbf{u}} = (u_1, \ldots, u_n)$ is a periodic point of period p, where p is a prime, of the SDS map $T = [K_n, f, \sigma]$. Write $T(\vec{\mathbf{u}}) = \vec{\mathbf{w}} = (w_1, \ldots, w_n)$. For each $k \in 1, 2, \ldots, n$, we have $u_k \neq w_k$, for each $j \in \{2, \ldots, p-1\}$, $T^j(\vec{\mathbf{u}}) \neq T(\vec{\mathbf{w}}) \neq \vec{\mathbf{u}}$ and $T^p(\vec{\mathbf{u}}) = T^{p-1}(\vec{\mathbf{w}}) = \vec{\mathbf{u}}$. In particular, if $A = \mathbb{F}_p$, then T can be taken as $T(\vec{\mathbf{u}}) = \vec{\mathbf{u}} + (\overrightarrow{\mathbf{p-1}})$.*

Note that if we let $\alpha = \alpha_1\alpha_2\ldots\alpha_k$ be a finite word over the alphabet $\mathbb{F}_p$, we say that a vector $(u_1, u_2, \ldots, u_n) \in \mathbb{F}_p^n$ contains the subsequence α if there exist indices $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \ldots < i_k$ and $u_{i_j} = \alpha_j$ for all $j \in \{1, 2, \ldots, k\}$.

Definition 2 Let α be a finite word over the alphabet $\mathbb{F}_p$. Define $S_n(\alpha)$ to be the set of vectors $\vec{\mathbf{u}} \in \mathbb{F}_p^n$ such that $\vec{\mathbf{u}}$ contains the subsequence α.

Definition 3 Let $n \geq 3$ and let $\widehat{\mathbb{F}}_3^n = \{(u_1, \ldots, u_n) \in \mathbb{F}_3^n : u_n = 0\}$. Define $\widehat{T}_n$ to be the undirected simple graph with

$$V(\widehat{T}_n) = \widehat{\mathbb{F}}_3^n, \qquad E(\widehat{T}_n) = \left\{\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\} \subseteq \widehat{\mathbb{F}}_3^n : \vec{\mathbf{u}} + \vec{\mathbf{v}} \in S_n(102)\right\}.$$

The following two lemmas link the graphs $\widehat{T}_n$ to our study of 3-cycles in the phase spaces of SDS defined over complete graphs.

Lemma 2 *Let $n \geq 3$ be an integer, and let $C = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_k\}$ be a clique of order k of $\widehat{T}_n$. There exists a map $g : \mathbb{F}_3^n \to \mathbb{F}_3$ such that each $\vec{\mathbf{u}}_i \in C$ is in a 3-cycle of $\Lambda([K_n, g, id])$. Moreover, no three distinct $\vec{\mathbf{u}}_i$, $\vec{\mathbf{u}}_j$, $\vec{\mathbf{u}}_p$ are contained in the same 3-cycle of $\Lambda([K_n, g, id])$.*

Proof For each $i \in \{1, 2, \ldots, k\}$, let $\vec{\mathbf{u}}_i = (c_{i_1}, c_{i_2}, \ldots, c_{i_n})$, where $c_{i_n} = 0$ by the definition of $\widehat{T}_n$. Let $g(\vec{\mathbf{c}}) = \vec{\mathbf{c}} + \vec{\mathbf{2}} = \vec{\mathbf{N}}$. Define the map $g : \mathbb{F}_3^n \to \mathbb{F}_3$ as follows:

i. If there are some i, l such that $\vec{\mathbf{a}} = (\overline{c_{i_1}}, \overline{c_{i_2}}, \ldots, \overline{c_{i_l}}, c_{i_{l+1}}, \ldots, c_{i_n})$, then let $g(\vec{\mathbf{a}}) = \overline{c_{i_{l+1}}}$.

ii. If there are some i, l such that $\vec{\mathbf{a}} = (c_{i_1}, c_{i_2}, \ldots, c_{i_l}, \overline{c_{i_{l+1}}}, \overline{c_{i_{l+2}}}, \ldots, \overline{c_{i_n}})$, then let $g(\vec{\mathbf{a}}) = c_{i_{l+1}}$.

iii. If there are some i, l, v such that $\vec{\mathbf{a}} = (c_{i_1}, c_{i_2}, \ldots, c_{i_l}, \overline{c_{i_{l+1}}}, \ldots, \overline{c_{i_v}}, c_{i_{v+1}}, \ldots, c_{i_n})$, then let $g(\vec{\mathbf{a}}) = \overline{c_{i_{v+1}}}$, where $v \neq l+1$.

iv. Otherwise, let $g(\vec{\mathbf{a}}) = 0$.

We first need to show that g is well-defined. To this end, we show that for any i, $j \in \{1, 2, \ldots, k\}$ and any $l, m \in \{0, 1, \ldots, n-1\}$ with $i \neq j$ or $l \neq m$, we have

$$(\overline{c_{i_1}}, \overline{c_{i_2}}, \ldots, \overline{c_{i_l}}, c_{i_{l+1}}, \ldots, c_{i_n}) \neq (\overline{c_{j_1}}, \overline{c_{j_2}}, \ldots, \overline{c_{j_m}}, c_{j_{m+1}}, \ldots, c_{j_n}) \quad (1)$$

$$(\overline{c_{i_1}}, \overline{c_{i_2}}, \ldots, \overline{c_{i_l}}, c_{i_{l+1}}, \ldots, c_{i_n}) \neq (c_{j_1}, c_{j_2}, \ldots, c_{j_l}, \overline{c_{j_{m+1}}}, \overline{c_{j_v}}, c_{j_{v+1}} \ldots, c_{j_n}) \quad (2)$$

$$(\overline{c_{i_1}}, \overline{c_{i_2}}, \ldots, \overline{c_{i_l}}, c_{i_{l+1}}, \ldots, c_{i_n}) \neq (c_{j_1}, c_{j_2}, \ldots, c_{j_l}, \overline{c_{j_{m+1}}}, \overline{c_{j_{m+2}}}, \ldots, c_{j_n}) \quad (3)$$

$$(c_{i_1}, c_{i_2}, \ldots, c_{i_l}, \overline{c_{i_{l+1}}}, \ldots, \overline{c_{i_v}}, c_{i_{v+1}}, \ldots, c_{i_n}) \neq (c_{j_1}, c_{j_2}, \ldots, c_{j_m}, \overline{c_{j_{m+1}}}, \ldots, \overline{c_{j_v}}, c_{j_{v+1}}, \ldots, c_{j_n}) \quad (4)$$

$$(c_{i_1}, c_{i_2}, \ldots, c_{i_l}, \overline{c_{i_{l+1}}}, \ldots, \overline{c_{i_v}}, c_{i_{v+1}}, \ldots, c_{i_n}) \neq (c_{j_1}, c_{j_2}, \ldots, c_{j_m}, \overline{c_{j_{m+1}}}, \overline{c_{j_{m+2}}}, \ldots, c_{j_n}) \quad (5)$$

$$(c_{i_1}, c_{i_2}, \ldots, c_{i_l}, \overline{c_{i_{l+1}}}, \overline{c_{i_{l+2}}}, \ldots, \overline{c_{i_n}}) \neq (c_{j_1}, c_{j_2}, \ldots, c_{j_m}, \overline{c_{j_{m+1}}}, \overline{c_{j_{m+2}}}, \ldots, c_{j_n}) \quad (6)$$

Also, if $i = j$ and $l \neq m$, then we obtain the same result with the Equations 1- 6. So, we can suppose that $i \neq j$. If equality holds in Eq. 1 or Eq. 6, then $\vec{\mathbf{u}}_i + \vec{\mathbf{u}}_j$ can be

$$(0, 0, \ldots, 0, 1, \ldots 1, 0, \ldots, 0), (1, 1, \ldots, 2, \ldots, 2, 1, \ldots 1),$$
$$(2, 2, \ldots, 2, 1, \ldots, 1, 2, \ldots, 2)$$

which are not in $S_n(102)$. But this is a contradiction. If equality holds in Eq. 2, then $\vec{\mathbf{u}}_i + \vec{\mathbf{u}}_j$ can be of the form

$$(0, 0, \ldots, 0, 1, \ldots 1, 0, \ldots, 0), (1, 1, \ldots, 1, 2, \ldots 2, 0, \ldots, 0),$$
$$(2, 2, \ldots, 2, 1, \ldots 1, 0, \ldots, 0)$$

which are not in $S_n(102)$. But this is a contradiction. If equality holds in Eq. 3, then $\vec{\mathbf{u}}_i + \vec{\mathbf{u}}_j$ can be of the form

$$(1, \ldots, 1, 0, \ldots, 0), (2, \ldots, 2, 0, \ldots, 0), (2, \ldots, 2, 1, \ldots, 1),$$
$$(0, \ldots, 0, 1, \ldots, 1), (1, \ldots, 1, 2, \ldots, 2), (0, \ldots, 0, 2, \ldots, 2)$$

which are not in $S_n(102)$. But this is a contradiction. If equality holds in Eq. 4, then $\vec{\mathbf{u}}_i + \vec{\mathbf{u}}_j$ can be of the form

$$(1, \ldots, 1, 0, \ldots, 0), (2, \ldots, 2, 0, \ldots, 0), (1, 1, \ldots, 1, 2, \ldots 2, 0, \ldots, 0),$$
$$(2, 2, \ldots, 2, 1, \ldots 1, 0, \ldots, 0), \quad (2, 2, \ldots, 2, 2, \ldots 2, 0, \ldots, 0)$$

which are not in $S_n(102)$. But this is a contradiction. If equality holds in Eq. 5, then $\vec{\mathbf{u}}_i + \vec{\mathbf{u}}_j$ can be of the form

$$(0, 0, \ldots, 0, 2, \ldots 2, 1, \ldots, 1), (1, 1, \ldots, 1, 2, \ldots 2, 1, \ldots, 1)$$

which are not in $S_n(102)$. But this is a contradiction. Hence, g is well-defined. Let $T = [K_n, g, id]$. We need to show that for each $i \in \{1, 2, \ldots, k\}$, the vector $\vec{\mathbf{u}}_i$ is in a 3-cycle of $\Lambda(T)$.

By the definition of g, for each $l \in \{1, 2, \ldots, n\}$ we have

$$G_l(\vec{\mathbf{u}}_i) = (\overline{c_{i_1}}, \overline{c_{i_2}}, \ldots, \overline{c_{i_l}}, c_{i_{l+1}}, \ldots, c_{i_n}),$$

and

$$G_l(\vec{\mathbf{u}}_i) = (\overline{c_{i_1}}, \overline{c_{i_2}}, \ldots, \overline{c_{i_l}}, \overline{c_{i_{l+1}}}, \ldots, \overline{c_{i_n}}) = (\widehat{c_{i_1}}, \widehat{c_{i_2}}, \ldots, \widehat{c_{i_l}}, \overline{c_{i_{l+1}}}, \ldots, \overline{c_{i_n}}),$$

where $\widehat{c_i} = c_i + 1$. Moreover,

$$G_l(\widehat{c_{i_1}}, \widehat{c_{i_2}}, \ldots, \widehat{c_{i_l}}, \widehat{c_{i_{l+1}}}, \ldots, \widehat{c_{i_n}}) = (c_{i_1}, c_{i_2}, \ldots, c_{i_l}, \widehat{c_{i_{l+1}}}, \ldots, \widehat{c_{i_n}}).$$

Furthermore,

$$T(\vec{\mathbf{u}}_i) = (\overline{u_1}, \overline{u_2}, \ldots, \overline{u_k}) \neq (u_1, u_2, \ldots, u_k),$$

$$T^2(\vec{\mathbf{u}}_i) = T(\overline{u_1}, \overline{u_2}, \ldots, \overline{u_k}) = \vec{\mathbf{u}}_i,$$

and

$$T^3(\vec{\mathbf{u}}_i) = T(T^2(\vec{\mathbf{u}}_i)) = T(\widehat{u_1}, \widehat{u_2}, \ldots, \widehat{u_k}) = \vec{\mathbf{u}}_i.$$

Finally, choose distinct indices $i,\ j,\ p \in \{1, 2, \ldots, k\}$. We claim that the vectors $\vec{\mathbf{u}}_i$, $\vec{\mathbf{u}}_j$, $\vec{\mathbf{u}}_p$ lie in distinct 3-cycles of $\Lambda(T)$. Since $\vec{\mathbf{u}}_i \neq \vec{\mathbf{u}}_j \neq \vec{\mathbf{u}}_p$, the only way they could belong to the same 3-cycle (without loss of generality) would be if

$$T(\vec{\mathbf{u}}_i) = \vec{\mathbf{u}}_j.$$

But we know

$$T(\vec{\mathbf{u}}_i) = (\overline{u_1}, \overline{u_2}, \ldots, \overline{u_k}),$$

so this would require $(\overline{u_1}, \overline{u_2}, \ldots, \overline{u_k}) = \vec{\mathbf{u}}_j$. This is impossible, since $\vec{\mathbf{u}}_i$ and $\vec{\mathbf{u}}_j$ both have the same last coordinate, which is fixed at 0. Hence, $\vec{\mathbf{u}}_i$, $\vec{\mathbf{u}}_j$ and $\vec{\mathbf{u}}_p$ belong to distinct 3-cycles of $\Lambda(T)$. □

Lemma 3 *Let $n \geq 3$ be an integer, and let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ be distinct nonadjacent vertices of $\widehat{T}_n$. Let $g : \mathbb{F}_3^n \to \mathbb{F}_3$ be a function. If $\vec{\mathbf{u}}$ lies on a 3-cycle of the phase space $\Lambda([K_n, g, id])$, then at least two of $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ do not lie on a 3-cycle of $\Lambda([K_n, g, id])$.*

Proof Assume, toward a contradiction, that two distinct states $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \Lambda([K_n, g, id])$ lie on (possibly different) 3-cycles of the phase space $\Lambda([K_n, g, id])$. Without loss of generality, we may assume both are contained in 3-cycles. Let $T = [K_n, g, id]$ and write $\vec{\mathbf{u}} = (a_1, a_2, \ldots, a_n)$ and $\vec{\mathbf{v}} = (b_1, b_2, \ldots, b_n)$, with all coordinates taken in $\mathbb{Z}_3$. By the construction, $a_n = b_n = 0$.

Let

$$\alpha = \min\{\, i : a_i \neq b_i \,\}, \beta = \max\{\, i : a_i \neq b_i\}.$$

Since $a_n = b_n = 0$, we have $\beta \leq n - 1$. As $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are not adjacent in $\widehat{T}_n$, we have

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (a_1 + b_1, \ldots, a_n + b_n) \notin S_n(102).$$

Consequently,

$$a_i + b_i = \begin{cases} 1 \text{ or } 2, & \text{if } \alpha \le i \le \beta, \\ 0, & \text{otherwise,} \end{cases}$$

so $b_i = a_i$ for $i \notin [\alpha, \beta]$ and $b_i \in \{\overline{a_i}, \overline{\overline{a_i}}\}$ for $\alpha \le i \le \beta$, where $\overline{x} := x+1 \pmod 3$.

By Lemma 1, each state on a 3-cycle satisfies $T(\vec{\mathbf{x}}) = \vec{\mathbf{x}} + \vec{2}$, where $\vec{2}$ is the all-2 vector. In particular, for every $1 \le \ell \le n$,

$$G_\ell(\vec{\mathbf{x}}) = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_\ell}, x_{\ell+1}, \ldots, x_n),$$

and Lemma 1 further gives the coordinate identities

$$g\big(G_\beta^2(\vec{\mathbf{u}})\big) = \overline{a_{\beta+2}}, \quad g\big(G_{\alpha-1}^2(\vec{\mathbf{v}})\big) = \overline{b_{\alpha+1}},$$
$$g\big(G_{\alpha-1}^2(\vec{\mathbf{u}})\big) = \overline{a_{\alpha+1}}, \quad g\big(G_\beta^2(\vec{\mathbf{v}})\big) = \overline{b_{\beta+2}}.$$

Since $b_i = a_i$ for $i \notin [\alpha, \beta]$, we have $b_{\beta+2} = a_{\beta+2}$ and hence $g(G_\beta^2(\vec{\mathbf{v}})) = \overline{a_{\beta+2}}$. Moreover, by Lemma 1 the expressions $g(G_{\alpha-1}^2(\cdot))$ and $g(G_\beta^2(\cdot))$ compute the same coordinate of $T^2(\cdot)$; therefore, for $\vec{\mathbf{u}}$ they agree, and we obtain

$$\overline{a_{\alpha+1}} = \overline{a_{\beta+2}}.$$

Combining this with $g(G_{\alpha-1}^2(\vec{\mathbf{v}})) = \overline{b_{\alpha+1}}$ and the fact that $a_{\alpha+1} \neq b_{\alpha+1}$ (since α is the first index of disagreement), we get

$$\overline{a_{\alpha+1}} = \overline{a_{\beta+2}} = g\big(G_\beta^2(\vec{\mathbf{v}})\big) = \overline{b_{\beta+2}} = \overline{a_{\beta+2}} \quad \text{and} \quad \overline{b_{\alpha+1}} = a_{\alpha+1}.$$

Thus $\overline{a_{\alpha+1}} = a_{\alpha+1}$, which is impossible in $\mathbb{F}_3$. This contradiction completes the proof. □

In the SDS setting considered here, the value θ_n denotes the maximum possible number of 3-cycles in the phase space on n vertices. The quantity $\psi(\widehat{T}_n)$ arises from a graph-theoretic construction, where $\widehat{T}_n$ encodes admissible state transitions that satisfy the period-3 constraint, and $\psi(\cdot)$ denotes the clique number. The key insight is that counting 3-cycles in the SDS can be reduced to finding the largest complete subgraph in $\widehat{T}_n$. This reduction transforms a dynamical systems problem into a purely combinatorial one, enabling the use of graph theory tools to compute or bound θ_n. Let $\psi(Y)$ denote the clique number of a graph Y.

Theorem 4 *For any integer $n \ge 3$, we have $\theta_n = \psi(\widehat{T}_n)$.*

Proof Let $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_{\theta_n}$ be vertices of $\widehat{T}_n$. Choose an integer $n \ge 3$, and let $k = \psi(\widehat{T}_n)$. Let $C = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_k\}$ be a clique of order k of $\widehat{T}_n$. By Lemma 2, there exists a map $g : \mathbb{F}_3^n \to \mathbb{F}_3$ such that any distinct vectors $\vec{\mathbf{u}}_i, \vec{\mathbf{u}}_j, \vec{\mathbf{u}}_p \in C$ are in distinct 3-cycles of $\Lambda([K_n, g, \mathrm{id}])$. Then, $\Lambda([K_n, g, \mathrm{id}])$ contains at least k 3-cycles. Also, we have $g \in \mathbb{F}_3^{\mathbb{F}_3^n}$ and $\theta(g, \mathrm{id}) \ge k$. So,

$$\theta_n = \max \theta(h, \mathrm{id}) \ge \theta(g, \mathrm{id}) \ge k.$$

By Definition 1, we know that there exists a function $h \in \mathbb{F}_3^{\mathbb{F}_3^n}$ such that $\theta(h, \mathrm{id}) = \theta_n$. This means that there are θ_n 3-cycles in the phase space of $\Lambda([K_n, g, \mathrm{id}])$. It follows from Lemma 1 that each 3-cycle of $\Lambda([K_n, g, \mathrm{id}])$ contains exactly one vector whose last coordinate is 0. In the notation of Definition 1, each of the θ_n 3-cycles of $\Lambda([K_n, g, \mathrm{id}])$ contains exactly one vector that is a vertex of $\widehat{T}_n$. Let $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_{\theta_n}$ be vertices of $\widehat{T}_n$. Choose some distinct $i, j, p \in \{1, 2, \ldots, \theta_n\}$. Because each of the vectors $\vec{\mathbf{u}}_i, \vec{\mathbf{u}}_j, \vec{\mathbf{u}}_p$ are in a 3-cycle of $\Lambda([K_n, g, \mathrm{id}])$, it follows from Lemma 3 that $\vec{\mathbf{u}}_i, \vec{\mathbf{u}}_j$ and $\vec{\mathbf{u}}_p$ must be adjacent in $\widehat{T}_n$. Since i, j and p were arbitrary, this shows that $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_{\theta_n}\}$ is a clique of $\widehat{T}_n$. Hence, $\theta_n \leq k$ and so $\theta_n = k$. □

In order to clarify the connection established in Theorem 4, we next present a concrete construction in the case $n = 4$. By explicitly exhibiting a clique in $\widehat{T}_4$ and defining a corresponding vertex function g according to Lemma 2, we show how the existence of such a clique guarantees the presence of disjoint 3-cycles in the phase space of the SDS.

Example 2 Let $n = 4$ and work over $\mathbb{F}_3$. Consider the complete graph K_4 and the update order $\sigma = 1234$. We will realize a 2 -clique in $\widehat{T}_4$ and build a vertex function $g : \mathbb{F}_3^4 \to \mathbb{F}_3$ as in Lemma 2 so that each vector in the clique lies on its own 3-cycle for the SDS map $T = [K_4, g, \sigma]$, with

$$T(\vec{\mathbf{u}}) = \vec{\mathbf{u}} + \vec{2}, \qquad T^2(\vec{\mathbf{u}}) = \vec{\mathbf{u}} + \vec{1}, \qquad T^3(\vec{\mathbf{u}}) = \vec{\mathbf{u}}.$$

Let

$$\vec{\mathbf{u}} = (0, 0, 0, 0), \qquad \vec{\mathbf{v}} = (1, 0, 2, 0).$$

Both have last coordinate 0, so $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \widehat{\mathbb{F}}_3^4$. Moreover,

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (1, 0, 2, 0)$$

contains the subsequence 102, hence $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ is an edge of $\widehat{T}_4$ i.e., $C = \{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ is a clique of order 2. For $i = 1, 2, \ldots, 4$ write $\overline{x}_i := x_i + 2$ and $\widehat{x}_i := x_i + 1$ in $\mathbb{F}_3$. For $i \in \{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ and $l \in \{0, 1, 2, 3\}$, set

$$\begin{cases} g(\overline{c_{i_1}}, \ldots, \overline{c_{i_l}}, c_{i_{l+1}}, \ldots, c_{i_4}) = \overline{c_{i_{l+1}}}, \\ g(c_{i_1}, \ldots, c_{i_l}, \overline{c_{i_{l+1}}}, \ldots, \overline{c_{i_4}}) = c_{i_{l+1}}, \\ g(c_{i_1}, \ldots, c_{i_l}, \overline{c_{i_{l+1}}}, \ldots, \overline{c_{i_v}}, c_{i_{v+1}}, \ldots, c_{i_4}) = \overline{c_{i_{v+1}}} \quad (v \neq l+1), \\ g(\mathbf{c}) = 0 \ \text{ otherwise.} \end{cases}$$

Let $G_\ell = L_\ell \circ \cdots \circ L_1$ be the partial update maps so $T = G_4$. For every $i \in \{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ and $\ell = 0, 1, 2, 3$ the rule (i) gives

$$g\big(G_\ell(i)\big) = \overline{c_{i_{\ell+1}}} \Longrightarrow T(i) = \big(\overline{c_{i_1}}, \overline{c_{i_2}}, \overline{c_{i_3}}, \overline{c_{i_4}}\big) = i + \vec{2},$$

so $T(\vec{\mathbf{u}}) = \vec{\mathbf{u}} + \vec{2}$ and $T(\vec{\mathbf{v}}) = \vec{\mathbf{v}} + \vec{2}$. Similarly, rules (ii)-(iii) force

$$T^2(\vec{\mathbf{b}}) = \vec{\mathbf{b}} + \vec{1}, \qquad T^3(\vec{\mathbf{b}}) = \vec{\mathbf{b}},$$

i.e., both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ lie on 3-cycles. By Lemma 3, since $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are distinct vertices of $\widehat{T_4}$ and g is built by Lemma 2, the 3-cycles through $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are distinct. Hence $\Lambda([K_4, g, \sigma])$ contains at least $|C| = 2$ disjoint 3-cycles. This example witnesses $\theta_4 \geq 2$ and realizes a clique of the same size in $\widehat{T_4}$, in accord with Theorem 4 which states $\theta_n = \psi(\widehat{T_n})$ when maximized over g.

Given an integer $n \geq 3$, we may define a map $\rho_n : \widehat{\mathbb{F}_3^n} \to \mathbb{F}_3^{n-1}$ by

$$\rho_n(u_1, u_2, \ldots, u_n) = (u_1, u_2, \ldots, u_{n-1}).$$

Also, if $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \widehat{\mathbb{F}_3^n}$, then $\vec{\mathbf{u}} + \vec{\mathbf{v}} \in S_n(102)$ if and only if $\rho_n(\vec{\mathbf{u}}) + \rho_n(\vec{\mathbf{v}}) \in S_{n-1}(102)$. Then, we can give the following definition.

Definition 4 Given an integer $m \geq 3$, define T_m to be the undirected simple graph with vertex set $V(T_m) = \mathbb{F}_3^m$ and edge set $E(T_m) = \{\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\} \subseteq \mathbb{F}_3^m : \vec{\mathbf{u}} + \vec{\mathbf{v}} \in S_m(102)\}$.

We know that ρ_{n+1} is a graph isomorphism from the graph $\widehat{T_{n+1}}$ to the graph T_n. So, $\theta_{n+1} = \psi(\widehat{T_{n+1}}) = \psi(T_n)$.

3 Relation to Codes

Let $A_3(n, 4)$ denote the maximum number of codewords that can appear in a ternary code of length n with minimum distance at least 4. Given an integer $m \geq 3$, define L_m to be the undirected simple graph with vertex set

$$V(L_m) = \mathbb{F}_3^m$$

and edge set

$$E(L_m) = \{\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\} \subseteq \mathbb{F}_3^m : \vec{\mathbf{u}} + \vec{\mathbf{v}} \in S_m(1111)\}.$$

Then $A_3(n, 4)$ equals the clique number $\psi(L_n)$ of the graph L_n. Each 3-cycle in the SDS phase space corresponds to a triple of distinct states in $\mathbb{F}_3^n$ whose pairwise Hamming distances are at least 4. This separation ensures that cycles remain disjoint under the SDS update order and mirrors the coding-theoretic property of a ternary code with minimum distance ≥ 4. Theorem 5 establishes a one-to-one correspondence between the set of disjoint 3-cycles in the SDS and the set of codewords in an optimal ternary $(n, 4)$-code. Therefore, known extremal bounds for $A_3(n, 4)$ directly yield the maximum possible number of 3-cycles in the SDS.

Theorem 5 *For any positive integer n,* $\theta_{n+1} = A_3(n, 4)$.

Proof It suffices to exhibit a graph isomorphism $F : L_n \to T_n$. There exists an invertible linear map $F : \mathbb{F}_3^n \to \mathbb{F}_3^n$ with the property that a vector contains the subsequence 1111 if and only if its image under F contains 102 one may obtain F by a suitable lower-triangular Toeplitz map implementing a fixed finite-impulse response filter over $\mathbb{F}_3$. Consequently, $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ is an edge of L_n if and only if $\{F(\vec{\mathbf{u}}), F(\vec{\mathbf{v}})\}$ is an

edge of T_n, i.e., F induces a graph isomorphism $L_n \cong T_n$. Hence $\psi(L_n) = \psi(T_n)$ and, by Theorem 4, $\theta_{n+1} = \psi(T_n) = \psi(L_n) = A_3(n, 4)$. □

As a further illustration, we examine the case $n = 5$. By identifying a 3-clique in $\widehat{T_5}$ and constructing an appropriate vertex function g in accordance with Lemma 2, we obtain three distinct 3-cycles in the phase space. This example shows how larger cliques in $\widehat{T_n}$ correspond to an increased number of periodic orbits, providing additional evidence for Theorem 4.

Example 3 Let $n = 5$ and work over $\mathbb{F}_3$. Consider the complete graph K_5 with update order $\sigma = 12345$. We construct a 3 -clique in $\widehat{T_5}$ and then build a vertex function $g : \mathbb{F}_3^5 \to \mathbb{F}_3$ as in Lemma 2 so that each vector in the clique lies on its own 3-cycle for $T = [K_5, g, \sigma]$. Define three states all with last coordinate 0.

$$\vec{\mathbf{u}} = (0, 2, 1, 0, 0), \qquad \vec{\mathbf{v}} = (1, 1, 1, 0, 0), \qquad \vec{\mathbf{w}} = (0, 2, 2, 2, 0).$$

Their pairwise sums are

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (1, 0, 2, 0, 0), \quad \vec{\mathbf{u}} + \vec{\mathbf{w}} = (0, 1, 0, 2, 0), \quad \vec{\mathbf{v}} + \vec{\mathbf{w}} = (1, 0, 0, 2, 0),$$

each of which contains the subsequence 102. Hence $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ is a 3-clique of $\widehat{T_5}$. By Lemma 2, there exists $g : \mathbb{F}_3^5 \to \mathbb{F}_3$ given by the casewise rules there, with $\overline{x}_i = x_i + 2$ and $\widehat{x_i} = x_i + 1$ such that, for each $\vec{\mathbf{x}} = (x_1, x_2, x_3, x_4, x_5) \in \{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\}$,

$$T(\vec{\mathbf{x}}) = \vec{\mathbf{x}} + \vec{2}, \qquad T^2(\vec{\mathbf{x}}) = \vec{\mathbf{x}} + \vec{1}, \qquad T^3(\vec{\mathbf{x}}) = \vec{\mathbf{x}},$$

and the three 3-cycles are pairwise distinct by the last clause of Lemma 2 and Lemma 3. Thus $\Lambda([K_5, g, \sigma])$ contains at least 3 disjoint 3-cycles, i.e., $\theta_5 \geq 3$. On the coding side, $A_3(4, 4) = 3$, so Theorem 5 predicts $\theta_5 = A_3(4, 4) = 3$.

4 Concluding Remarks

In this study, we have partially generalized the results of the paper [1] and we have settled one of the direction given in there. In Theorem 5, we have found the result for the maximum number of codewords that can appear in a ternary code of length n that has minimum distance at least 4. An interesting direction for further research is to generalize the linear transformation to the case of arbitrary m instead of 4.

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Declarations

Conflicts of Interest All authors declare that they have no conflict of interest.

References

1. Defant, C.: Binary codes and period-2 orbits of sequential dynamical systems. Discrete Mathematics and Theoretical Computer Science **19**, (Combinatorics) (2017)
2. Mortveit, H., Reidys, C.: An Introduction to Sequential Dynamical Systems. Springer, New York (2007)
3. Barrett, C.L., Mortveit, H.S., Reidys, C.M.: Elements of a theory of simulation II: sequential dynamical systems. Appl. Math. Comput. **107**(2–3), 121–136 (2000)
4. Barret, C., Mortveit, H., Reidys, C.: Elements of a theory of computer simulation III. Appl. Math. Comput. **122**, 325–340 (2002)
5. Barrett, C.L., Reidys, C.M.: Elements of a theory of computer simulation I: sequential CA over random graphs. Appl. Math. Comput. **98**(2–3), 241–259 (1999)
6. Garcia, L.D., Jarrah, A.S., Laubenbacher, R.: Sequential dynamical systems over words. Appl. Math. Comput. **174**(1), 500–510 (2006)
7. Read, R.: The number of k-coloured graphs on labelled nodes. Can. J. Math. **12**, 410–414 (1960)
8. Aledo, J.A., Diaz, L.G., Martinez, S., Valverde, J.C.: On periods and equilibria of computational sequential systems. Inf. Sci. **409**, 27–34 (2017)
9. Chen, W.Y., Li, X., Zheng, J.: Matrix method for linear sequential dynamical systems on digraphs. Appl. Math. Comput. **160**(1), 197–212 (2005)
10. Reidys, C.: Acyclic orientations of random graphs. Adv. Appl. Math. **21**(2), 181–192 (1998)
11. Savage, C.D., Zhang, C.-Q.: The connectivity of acyclic orientation graphs. Discret. Math. **184**(1–3), 281–287 (1998)
12. West, D.B.: Acyclic orientations of complete bipartite graphs. Discret. Math. **138**(1–3), 393–396 (1995)
13. Galay, R., Kalistratov, D.: A new algebraic approach to the graph isomorphism and clique problems. arXiv:1905.08033 (2019)
14. Davis, B.: Unlabelling signed graph colorings and acyclic orientations. Master Thesis, San Francisco State University, https://matthbeck.github.io/teach/masters/brian.pdf (2014)
15. Berlov, S.: Relationships between the clique number, chromatic number, and the degree for some graphs. Autom. Control. Comput. Sci. **44**(7), 407–414 (2010)
16. Kwiatkowski, M., Pankov, M.: On maximal cliques in the graph of simplex codes. J. Geom. **115**(1), 10 (2024)
17. Macauley, M., Mortveit, H.S.: Cycle equivalence of graph dynamical systems. Nonlinearity **22**(2), 421 (2009)
18. Laubenbacher, R., Pareigis, B.: Equivalence relations on finite dynamical systems. Adv. Appl. Math. **26**(3), 237–251 (2001)
19. Barrett, C.L., Mortveit, H.S., Reidys, C.M.: ETS IV: sequential dynamical systems: fixed points, invertibility and equivalence. Appl. Math. Comput. **134**(1), 153–171 (2003)
20. Mallik, S., Yildiz, B.: Graph theoretic aspects of minimum distance and equivalence of binary linear codes. Australas. J Comb. **79**, 515–526 (2021)
21. Klimov, A.B., Munoz, C., Sánchez-Soto, L.L.: Graph states in phase space. J. Phys. A: Math. Theor. **45**(21), 215303 (2012)
22. Mowshowitz, A., Dehmer, M., Emmert-Streib, F.: A note on graphs with prescribed orbit structure. Entropy **21**(11), 1118 (2019)
23. Van Lint, J.H., Wilson, R.M.: A Course in Combinatorics. Cambridge University Press, New York (2001)
24. Ling, S., Xing, C.: Coding Theory: A First Course. Cambridge University Press, New York (2004)
25. Milshtein, M.: A new binary code of length 16 and minimum distance 3. Inf. Process. Lett. **115**(12), 975–976 (2015)
26. Bridoux, F., Durbec, A., Perrot, K., Richard, A.: Complexity of fixed point counting problems in Boolean networks. J. Comput. Syst. Sci. **126**, 138–164 (2022)
27. Wu, S., Adiga, A., Mortveit, H.S.: Limit cycle structure for dynamic bi-threshold systems. Theoret. Comput. Sci. **559**, 34–41 (2014)

28. Solé, P., Zaslavsky, T.: The covering radius of the cycle code of a graph. Discret. Appl. Math. **45**(1), 63–70 (1993)
29. Alon, N., Yuster, R., Zwick, U.: Finding and counting given length cycles. Algorithmica **17**(3), 209–223 (1997)
30. Barrett, C., Hunt, M., Marathe, M., Ravi, S., Rosenkrantz, D., Stearns, R., Tosic, P.: Gardens of Eden and fixed points in sequential dynamical systems. Discrete Mathematics & Theoretical Computer Science (Proceedings) (2001)
31. Diestel, R.: Graph Theory, vol. 173. Springer, Berlin (2025)
32. West, D.B.: Introduction to Graph Theory, vol. 2. Prentice Hall, Upper Saddle River, NJ (2001)

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